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EXTENDED ABSTRACTS

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Dynamics of a spin-transfer nanooscillator controlled by external magnetic field.

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Spin-transfer nanooscillators (STNO) are promising nanoscale devices for applications in modern high-tech electronics and could be exploited in a different ways, for instance: as extremely small ultra-high frequency generators with central frequency from 10 to 100 GHz and beyond; as a trivial memory cell with two states and much more. In current work we study dynamical regimes of STNO with nanopillar unidomain structure under external magnetic field impact and their mutual transitions controlled by variations of control parameters. One can find that with a certain values of control parameters the STNO could function in autooscillatory regime; in stable state with constant magnetization; in multistable states of different types: stable/stable and stable/autooscillatory.

Chimera state realization in chaotic systems. The role of hyperbolicity

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In the present talk the transition from spatial coherence regimes to a noncoherence regime is analyzed in ensembles of discrete and continuous chaotic systems with nonlocal couplings. The possibility of existence of so-called "chimera" states is studied for a ring of nonlocally coupled identical chaotic oscillators. We consider nonhyperbolic systems which include maps with a single quadratic extremum, differential systems with Shil'nikov's type attractor as well as a class of hyperbolic systems which enclose the Lozi map and the Lorenz system. We hypothesize that two basic models, i.e., the Henon map and the Lozi map, can be used to generalize the implementation of chimera states to a sufficiently wide calss of chaotic systems. The Henon map can be served as an example of a partial system to describe the realization of chimera states in a ring of one-dimensional Feigenbaum's type maps, two-dimensional maps with period-doublings as well as differential systems with a saddle-focus separatrix loop according to Shil'nikov. The Lozi map can model the dynamics of ensembles of Lorenz-type oscillators. Our hypothesis consists in that the chimera states can be observed in ensembles of nonhyperbolic oscillators which can typically demonstrate the effect of multistability. The chimera modes cannot exist in ensembles whose partial element is represented by a hyperbolic system. Our suggestion is confirmed by the results of the previously published works.

In the talk we describe the numerical simulation results for the transition "hyperbolicity – nonhyperbolicity" in a ring of nonlocally coupled Lorenz oscillators. It is shown that chimera states cannot be realized in the hyperbolic regime. However, if the parameters of a partial Lorenz system are chosen in the nonhyperbolic region, the chimera states can be confidently observed. These results can be considered as a further confirmation of the proposed hyporthesis.

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Singularly Hyperbolic Attractors in Phase Systems

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An attractor is singularly hyperbolic if it is hyperbolic in terms of invariant cones' existence at each point of a versal parameter interval and the structure of the attractor changes at a set of bifurcational points within this parameter interval.

In this talk, we consider two types of phase systems in discrete and continuous type and prove the existence of singularly hyperbolic attractors. The first, discrete-time system is

$$\begin{aligned} &x(i+1) = Ax(i) + Bf(\varphi(i)), \\ &\varphi = C^T x, \end{aligned} \tag{1}$$

where $x(i) \in R^n$, $f(\varphi(i))$ is a 2π-periodic scalar function, A, B, C are constant matrices. We use the comparison method to derive the conditions for the existence of a singular hyperbolic attractor in this system.

The second, continuous-time system, describing the interaction between oscillators of different types, reads

$$\begin{aligned} &\ddot{\varphi} + f(\varphi, \dot{\varphi})\dot{\varphi} + g(\varphi) = -\mu\ddot{y} \\ &\ddot{y} + h\dot{y} + y = -\mu\ddot{\varphi}, \end{aligned} \tag{2}$$

where $y \in R^1$, $\varphi \in S^1$, f and g are 2π-periodic functions. A representative example of this general phase system is oscillators connected via the Huygens's coupling [1,2]. For the general system (2) as well as for the Huygens-type system [1], we prove the existence of a wild attractor which belongs to the class of singularly hyperbolic attractors and has both oscillatory and rotatory trajectories.

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Repulsive Inhibition Promotes Synchrony in Excitatory Neurons: Help from the Enemy

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We show that the addition of pairwise repulsive inhibition to excitatory networks of bursting neurons induces synchrony, in contrast to one's expectations [1,2]. Through stability analysis, we reveal the mechanism of this purely synergistic phenomenon and demonstrate that the inhibition leads to the disappearance of a homoclinic bifurcation that governs the type of synchronized bursting. As a result, the inhibition causes the transition from square-wave to easier-to-synchronize plateau bursting, so that weaker excitation is sufficient to induce bursting synchrony. This effect is generic and observed in different models of bursting neurons and fast synaptic interactions. We also find a universal scaling law for the synchronization stability condition for large networks in terms of the number of excitatory and inhibitory inputs each neuron receives, regardless of the network size and topology.

We dedicate this talk to the memory of Leonid P. Shilnikov, the pioneer of homoclinic bifurcation theory, and emphasize the importance of homoclinic bifurcations for understanding the onset of synchronization in bursting networks.

This work was supported by the National Science Foundation under Grant No. DMS-1009744, the US ARO Network Sciences Program, and GSU Brains & Behavior program.

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The destruction of conservative dynamics in the system of phase equations by symmetry breaking

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It is well known [1,2] that if the equations for reversible dissipative flow system have a specific symmetry (i.e. they are invariant relatively to a coordinate transformation), the conservative dynamics can appear in this system. For example, this phenomenon occurs in the system of phase equations for a chain of coupled oscillators [3]:

$$\dot{\psi}_k = \Delta_k + \varepsilon f(\psi_{k-1} + \varepsilon f(\psi_{k+1} - 2\varepsilon f(\psi_k),$$

where ψ_k is the difference of phases of neighbor oscillators and $f(\psi)$ is the coupling function. In the case of four oscillators the system is invariant relatively to the involution

$\psi_k \to \pi - \psi_{n-k}$ if coupling function contains only odd Fourier harmonics, e.g. $f(\psi) = \sin\psi + A\sin 3\psi$ as it was taken in [3].
We study the effect of symmetry breaking on the dynamics of this system. We consider $f(\psi) = \sin\psi + (A-d)\sin 3\psi + d\sin 2\psi$ as the coupling function so d is the symmetry breaking parameter.
To investigate the system we obtained numerically the Poincare map with the symmetric plane $\psi_2 = \pi/2$ as a section plane. We reveal that the stable cycles of different periods appear with the increase of the parameter d but most of them exist in narrow band of parameter d values. We obtain the stable and unstable manifolds of saddle cycles and show that homo- and heteroclinic structures exist.

Also we reveal that the quasistable set of complex structure appears in some band of parameter d values. The point moves along this set during a thousands of iterations but finally comes to the stable cycle.

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A study of the stochastic resonance as a random dynamical system

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We study a standard model for the stochastic resonance from the point of view of dynamical systems. We present a framework for random dynamical systems with nonautonomous deterministic forcing and we prove the existence of an attracting random periodic orbit for a class of one-dimensional systems with a time-periodic component. In the case of the stochastic resonance, we can derive an indicator for the resonant regime.

The Lorenz system near the loss of the foliation condition

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The well-known Lorenz system is classically studied via its reduction to the one-dimensional Lorenz map, which captures the full behaviour of the dynamics of the system. The reduction requires that the stable and unstable foliations on the classic Poincaré section are transverse locally near the chaotic attractor. We study a parameter regime where this so-called foliation condition fails for the first time and, subsequently, the Lorenz map no longer accurately represents the dynamics. Specifically, we investigate the development of hooks in the Poincaré return map that marks the loss of the foliation condition. To this end, we study how the three-dimensional phase space is organised by the global invariant manifolds of saddle equilibria and saddle periodic orbits, where we make extensive use of the continuation of orbit segments formulated by a suitable two-point boundary value problem (BVP). In particular, we compute the intersection curves of the two-dimensional unstable manifold $W^u(\Gamma)$ of a periodic orbit Γ with the Poincaré section. We identify when hooks form in the Poincaré map by formulating as a BVP the point of tangency between $W^u(\Gamma)$ and the stable foliation. This approach allows us to continue this tangency accurately and efficiently in each pair of the three system parameters. In this way, we identify the conic region in parameter space where the classical Lorenz attractor exists. As is expected from earlier work by Bykov and Shilnikov, a curve of T-points lies on the bounding surface, and we show that it ends in a codimension-three T-point-Andronov-Hopf bifurcation point.

Title Diffusion through non-transverse heteroclinic chains: A long-time instability for the NLS

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We introduce a new mechanism for instability (diffusion) in dynamical systems, based in the shadowing of a sequence of invariant tori connected along *non-transverse* heteroclinic orbits, under some geometric restrictions. This mechanism can be readily applied to systems of large dimensions, like infinite-dimensional Hamiltonian systems, particularly the Nonlinear Schrödinger Equation with cubic defocusing.

This is a joint work with Adrià Simon and Piotr Zgliczcynski.

Strange Nonchaotic Attractor of Hunt and Ott Type in a System with Ring Geometry

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The objective of this report is to introduce a physically realizable system with robust strange nonchaotic attractor (SNA).

Hunt and Ott offered an artificial system in which presence of the robust SNA is proven [1]. An example of some real system was suggested by Kuznetsov and Jalnine [2]. In contrast to the work of Kuznetsov and Jalnine, where the scheme was composed of self-oscillating elements, the system presented here is a ring system of two damping oscillators with additional elements introducing amplification and nonlinearity. The model is similar to that suggested in Ref. [3], but differs in the phase transition rules between the alternately exciting oscillatory subsystems.

Let the natural frequency of the first oscillator be ω_0 and that of the second oscillator be $2\omega_0$. The model equations of the system are:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \varepsilon \frac{d}{dt} y \sin x(\omega_0 t + \theta), \tag{3}$$

$$\ddot{y} + \gamma\dot{y} + 4\omega_0^2 y = \varepsilon \frac{d}{dt} \frac{x}{\sqrt{1+x^2}} g(t), \tag{4}$$

$$\dot{\theta} = \frac{2\pi\omega}{T}, \omega = \frac{\sqrt{5}-1}{2}. \tag{5}$$

Here x and y are the generalized coordinates of the oscillatory subsystems, γ is a damping factor, ε is a small parameter, α is an amplification factor. The function $g(t)$ describes the external signal, being switched on for short time intervals τ with the period $\frac{2\pi N}{\omega_0}$, where N is integer:

$$g(t) = \begin{cases} a^2, & nT \le t < \tau, \\ 0, & nT + \tau \le t < (n+1)T. \end{cases} \tag{6}$$

Numerical simulation confirms that the evolution of the first oscillator phase is described by the map of Hant and Ott type. Some results of are presented in Figure 1.

Figure 1a shows the phase portrait of the system in the three-dimensional state space. Figure 1b illustrates the basic topological characteristic of the phase transfer. The vertical axis corresponds to phase of the first oscillator and the horizontal axis to the phase θ of external force with incommensurate frequency ω. The curve C_1 which turns around the torus in the θ direction is transformed by the map effect to the curve C_2, which makes one turn around the meridian and one turn around the parallel of the torus. The form of these curves just corresponds to topological nature of the map for the Hunt and Ott model.

Behavior of the system also is illustrated by Fourier spectrum shown in Figure 1c.

The dynamics of this system was explored in a quite wide range of parameters and manifests robustness analogous to that of the artificial model of Hunt and Ott.

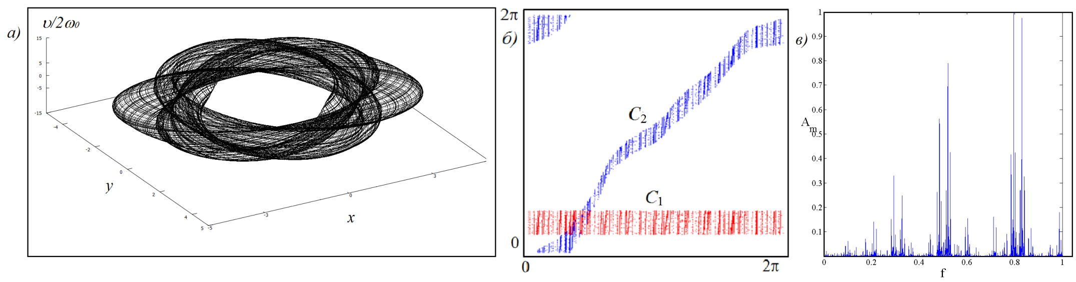

Figure 1. Phase portrait of the system in three-dimensional space (a). Numerical illustration of the basic topological properties of the phase transfer on the plot of φ versus θ (b). Fourier spectrum of signal produced by the system (c). Parameters: $\omega_0 = 6\pi, \tau = 3, \gamma = 0.25, a = 5$.

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Concept of Stability as a Whole of a Family of Fibers Maps for C^1-Smooth Skew Products and Its Generalization

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Let $F : I \to I$ be a skew product of maps of an interval, i.e. a dynamical system

$$F(x,\, y) = (f(x),\, g_x(y)), \quad \text{where} \quad g_x(y) = g(x,\, y), \ (x;\, y) \in I.$$

Here $I = I_1 \times I_2$ is a closed rectangle in the plane, I_1, I_2 are closed intervals.

In the framework of generalization of the concept of stability as a whole of a family of fibers maps for a C^1-smooth skew product of maps of an interval (see [1]) we introduce here the concept of dense stability as a whole of a family of fibers maps for C^1-smooth skew products of maps of an interval with a complicated dynamics of a quotient map.

Let $\widetilde{T}^1_*(I)$ be the space of C^1-smooth skew products of maps of an interval satisfying inclusion $F(\partial I) \subset \partial I$ for every $F \in \widetilde{T}^1_*(I)$ (here $\partial(\cdot)$ is a boundary of a set) with Ω-stable (in C^1-norm) quotient map. Let η^F_n and $\overline{\eta}^F_n$ $(n \geq 1)$ be auxiliary and suitable multifunctions respectively corresponding to n-th iteration of F (for definitions see [2]).

As it is proved in [2], the subspace of $\widetilde{T}^1_*(I)$ consisting of skew products with quotients of type $\succ 2^\infty$ (i.e. with quotients having periodic orbits with periods $\notin \{2^i\}_{i\geq 0}$)) can be presented as the union of 4 nonempty pairwise disjoint subspaces $\widetilde{T}^1_{*,j}(I)$ for $j = 1, 2, 3, 4$.

The above subspaces are distinguished in accordance with continuity property of auxiliary multifunctions (as $\widetilde{T}^1_{*,1}(I)$) or in accordance with continuity property of suitable (but not auxiliary) multifunctions (as $\widetilde{T}^1_{*,2}(I)$); or vise versa, in accordance with discontinuity property of suitable multifunctions in combination with continuity property of the Ω-function (as $\widetilde{T}^1_{*,3}(I)$) or in combination with discontinuity property of the Ω-function (as $\widetilde{T}^1_{*,4}(I)$). The Ω-function of a map $F \in \widetilde{T}^1_*(I)$ is the multifunction having graph in I which coincides with the nonwandering set of F.

Definition. A family of fibers maps of a skew product $F \in \widetilde{T}^1_{*,j}(I)$ $(j = 1, 2, 3, 4)$ *is densely stable as a whole in C^1-norm* if there is an open set $A(f) \subset I_1$ such that $A(f) \bigcap \Omega(f)$ is a proper, everywhere dense subset of f-nonwandering set $\Omega(f)$ with the following property: for any $\delta > 0$ one can find a neighborhood $B^1_\varepsilon(F)$ of F in $\widetilde{T}^1_*(I)$ such that for every map $\Phi \in B^1_\varepsilon(F) \bigcap \widetilde{T}^1_{*,j}(I)$ and for every recurrence time l^*_i $(i \geq i^*$ for some $i^* > i_*)$ of trajectories of the perfect part of f-nonwandering set $\Omega(f)$ there exists δ-close to the identity map in C^0-norm homeomorphism $H^{<l^*_i>} : \overline{\eta}^F_{l^*_i}|_{A(f)} \to \overline{\eta}^\Phi_{l^*_i}|_{A(\varphi)}$

$(H^{<l_i^*>} = (h_1(x),\, h_{2,x}^{<l_i^*>}(y)))$ satisfying the equalities

$$h_{2,x}^{<l_i^*>}{}_{|\overline{\eta}^F_{l_i^*\,|A(f)}(x)} \circ g_{x,l_i^*\,|\overline{\eta}^F_{l_i^*\,|A(f)}(x)}(y) =$$
$$\psi_{h_1(x),l_i^*\,|\overline{\eta}^\Phi_{l_i^*\,|A(\varphi)}(h_1(x))} \circ h_{2,x}^{<l_i^*>}{}_{|\overline{\eta}^F_{l_i^*\,|A(f)}(x)}(y),$$

where $(x,\, y) \in I$ is a point of the graph of a function $\overline{\eta}^F_{l_i^*\,|A(f)}$; $g_{x,l_i^*} : I_2 \to I_2$ for every $x \in I_1$, is a fiber map for $F^{l_i^*}$.

Skew products with densely stable as a whole in C^1-norm families of fibers maps exist in every space $\widetilde{T}^1_{*,j}(I)$ $(j = 1, 2, 3, 4)$.

Theorem. *Let $F \in \widetilde{T}^1_{*,j}(I)$ $(j = 1, 3$ or $4)$ be a skew product with densely stable as a whole family of fibers maps such that for every locally maximal quasiminimal set $K(f)$ of quotient f and for every $i \geq i^*$ there exists a connected component $C_{K(f),i}$ of the space of C^1-smooth Ω-stable maps of interval I_2 into itself satisfying*

$$\{g_{x,l_i^*}\}_{x\in K(f)} \subset \overline{C}_{K(f),i},$$

where $\overline{(\cdot)}$ is the closure of a set.
Then, F can be approximated up to any accuracy with use of C^1-smooth Ω-stable skew products of maps of an interval.

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Variety of strange pseudohyperbolic attractors in three-dimensional generalized Hénon maps

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We propose a sufficiently simple method, *the method of saddle charts*, for finding homoclinic attractors (here, strange attractors with saddle fixed points). This method is based on a preliminary detection of parameter domains related to the existence of fixed points with a given set of multipliers, e.g. such that a potential strange attractor with this point can be of Lorenz type. The obtained conditions are, in fact, only necessary ones, however, they simplify essentially a procedure of finding homoclinic attractors. We apply the method of saddle charts for three-dimensional generalized Hénon maps(GHM) for which we find “a zoo” of homoclinic attractors of various types.

An example of the saddle chart related to the zero fixed point O for GHM of form $\bar{x} = y, \bar{y} = z, \bar{z} = Bx + Az + Cy + f(y,z)$, where A, B, C are parameters (B is the Jacobian of the map), $f(0,0) = f'_y(0,0) = f'_z(0,0) = 0$, is shown in Fig.1 for $B = 0.5$. Here, on the (A,C)-parameter plane, 7 specific curves are shown: three lines, L^+, L^- and L^φ are bifurcation ones (the point O has multipliers $+1$, -1 and $e^{\pm i\varphi}$, respectively); the lines $AC+B = 0, A < 0$ as well as $S+$ and S^- corresponds to the primary resonances $\lambda_1 = -\lambda_2$, $\lambda_1 = \lambda_2 > 0$ and $\lambda_1 = \lambda_2 < 0$, respectively, between a pair of multipliers λ_1, λ_2 of the point O; finally, the line $C = 1 + B^2 + AB$ corresponds to the existence of the saddle O with the multipliers such that $\lambda_1\lambda_2 = -1$ and $\lambda_3 = -B$. Note that the line $C = B^2 - 1 - AB$ (containing L_φ) corresponds to the existence of the saddle O with $\lambda_1\lambda_2 = 1$ and $\lambda_3 = B$. These 7 lines divide the (A,C)-parameter plane onto several regions corresponding to different types of the point O that is a saddle always except for values of (A,C) from the dashed triangle domain, where the point is asymptotically stable.

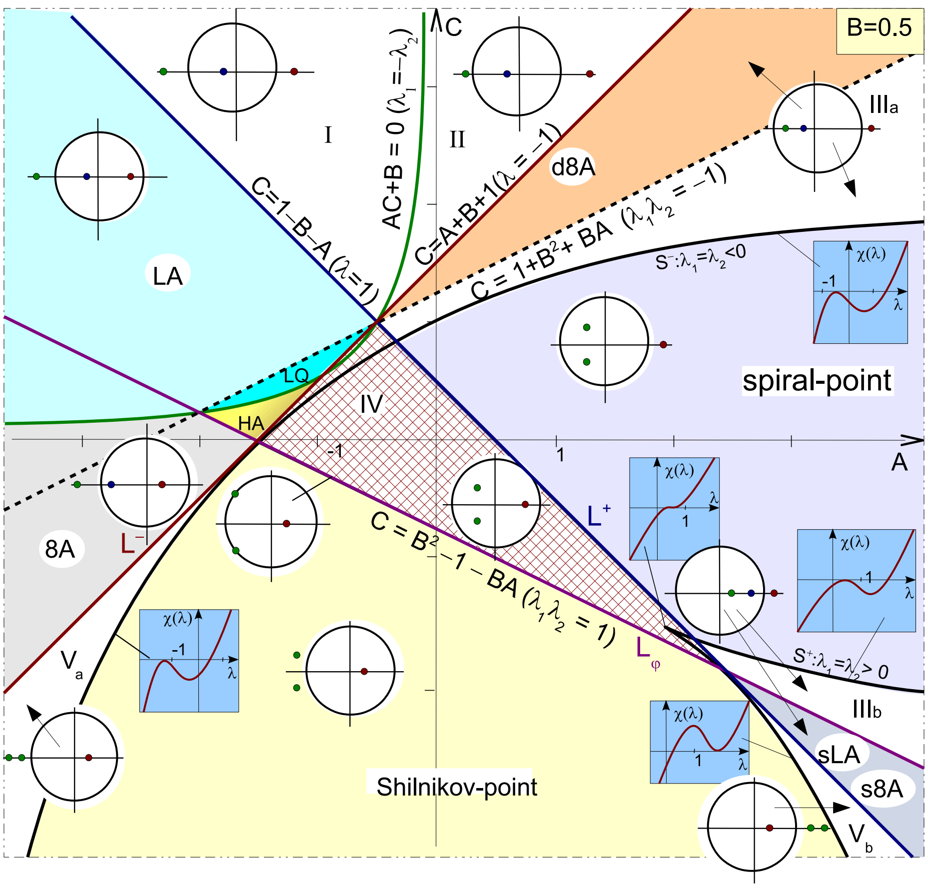

Figure 1: “Saddle chart” for 3-d generalized Henon map

The saddle chart method can be explained by means of figures below. So, in Fig. 2(I) we show a fragment of the saddle chart over the map of Lyapunov exponents for the corresponding GHM, see fig.(a). As usually, the Lyapunov map reflects in color an information on type of attractor (green color (1) corresponds to periodic attractors, turquoise one (2) – to invariant curves, yellow (3), red (4), blue (5) – to strange attractors with different spectra $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of Lyapunov exponents: in all cases $\Lambda_1 > 0, \Lambda_3 < 0$ and, besides, $\Lambda_2 < 0$ for domain (3); $\Lambda_2 \approx 0$ (up to numeric accuracy) for domain (4); $\Lambda_2 > 0$ for domain (5). The deep grey color (6) corresponds to homoclinic attractors containing the point O. In Fig. 2(I)(a), we see that the domain (6) belongs to the region LA of the saddle chart. It means that the corresponding map has a discrete Lorenz attractor, see Fig. 2(I)(c), containing the saddle O with multipliers (b).

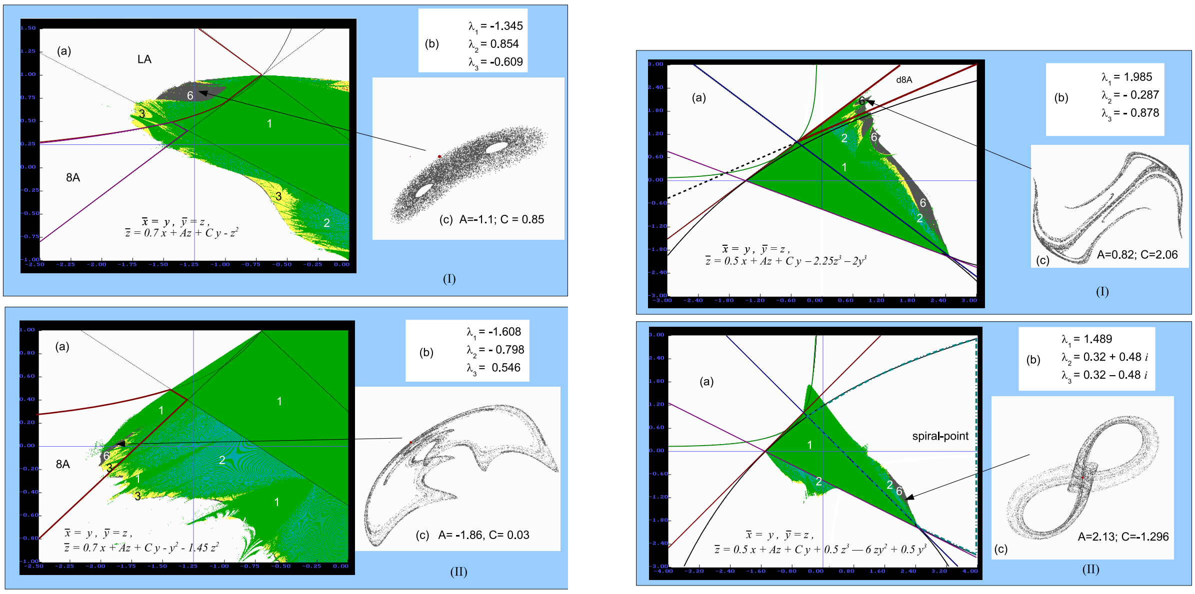

Figure 2:

Analogously, in Figures 2 and 3, there are illustrated the cases when the corresponding GHMs have discrete strange attractors of the following types: a figure-8 attractor, Fig. 2(II); a double figure-8 attractor, Fig. 3(I); a spiral attractor, Fig. 3(II).

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Reversible mixed dynamics

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Preamble

We say that a system possesses a *mixed dynamics* if

1) it has infinitely many hyperbolic periodic orbits of all possible types (stable, unstable,saddle) and

2) the closures of the sets of orbits of different types have nonempty intersections.

Recall that Newhouse regions are open domains (from the space of smooth dynamical systems) in which systems with homoclinic tangencies are dense. Newhouse regions in which systems with mixed dynamics are generic (compose residual subsets) are called *absolute Newhouse regions* or *Newhouse regions with mixed dynamics.* Their existence was proved in the paper [1] for the case of 2d diffeomorphisms close to a diffeomorphism with a nontransversal heteroclinic cycle containing two fixed (periodic) points with the Jacobians less and greater than 1. Fundamentally, that "mixed dynamics" is the universal property of reversible chaotic systems. Moreover, in this case generic systems from absolute Newhouse regions have infinitely many stable, unstable, saddle and symmetric elliptic periodic orbits [2,3].

As well-known, reversible systems are often met in applications and they can demonstrate a chaotic orbit behavior. However, the phenomenon of mixed dynamics means that this chaos can not be associated with "strange attractor" or "conservative chaos". Attractors and repellers have here a nonempty intersection containing symmetric orbits (elliptic and saddle ones) but do not coincide, since periodic sinks (sources) do not belong to the repeller (attractor). Therefore, "mixed dynamics" should be considered as a new form of dynamical chaos posed between "strange attractor" and "conservative chaos".

These and related questions are discussed in the talk. Moreover, the main attention here is paid to the development of the concept of mixed dynamics for two-dimensional reversible maps. The main elements of this concept are presented in section below.

Towards conception of mixed dynamics for two-dimensional reversible maps

A dynamical system is called *reversible* if it is invariant under the time reversal $t \to -t$ and a certain change of phase coordinates. In the most interesting and important cases, the latter coordinate change is an *involution*, i.e., such a diffeomorphism R of the phase space that $R^2 = \mathrm{Id}$. In the case of discrete dynamical systems, maps, one says that a map f is reversible, if the maps f and f^{-1} are conjugated by R, i.e., the following diagrams

$$\begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ \downarrow R & & \downarrow R \\ R(x) & \xrightarrow{f^{-1}} & f^{-1}(Rx) \end{array} \tag{7}$$

are commutative and, hence, the relations $Rf = f^{-1}R$ and $Rf^n = f^{-n}R$ hold for all points of the phase space.

It follows

that $(Rf^n)f = f^{-1}(f^{-n}R)$, i.e.

the maps f and f^{-1} are also conjugated by the diffeomorphisms Rf^n for any n which are involutions as well (indeed, one has $(Rf^n)^2 = Rf^n(Rf^n) = Rf^n(f^{-n}R) = R^2 = Id$).

The property of reversibility implies strong symmetries for the set of orbits, and here the set $Fix(R)$ of fixed points of the involution, i.e. a set of points x of the phase space such that $R(x) = x$, plays an important role. An orbit intersecting the set $Fix(R)$ (or the set $Fix(Rf^n)$) is called *symmetric*. Any symmetric periodic orbit possesses the following property: if it has a multiplier λ, then λ^{-1} is also its multiplier. Indeed, if $x_0 \in Fix(R)$ and $x_0 = f^n(x_0)$ for some natural n, then $R(x_0) = x_0$ and, hence, we obtain from diagram (7) that $x_0 = f^{-n}(x_0)$.

In the case of two-dimensional reversible maps, a symmetric periodic orbit has multipliers λ and λ^{-1}, and it is robust, if $\lambda \neq \pm 1$. Moreover, such an orbit with multipliers $e^{\pm i\varphi}$, where $\varphi \neq 0, \pi$, is, essentially, elliptic one, since the principal hypotheses of the KAM-theory hold here [4]. These properties make reversible and conservative systems to be related.

Concerning non-symmetric orbits, whose points, by definition, do not intersect the set $Fix(R)$ (as well as the sets $Fix(Rf^n)$ for any n), they can be, in principle, of arbitrary types. This property of reversible systems makes them to be related to systems of general type. However, for any non-symmetric orbit, there exists a symmetric to it orbit with "opposite" dynamical properties. It means that if a periodic orbit has multipliers λ_i, then the symmetric to it orbit will have multipliers λ_i^{-1}. Indeed, let x_0 be a periodic point for f, i.e. $x_0 = f^n(x_0)$. Then the point $y_0 = R(x_0)$, by the commutativity of diagrams (7), will be periodic for f^{-1}, i.e. $y_0 = f^{-n}(y_0)$. We say that symmetric each other orbits compose a *symmetric couple* of orbits.

The same as for dissipative case [5-8], in the space of reversible systems, open regions (Newhouse regions), in which reversible systems with both symmetric and non-symmetric homoclinic tangencies are dense, exist near any system with a symmetric homoclinic tangency. The proof of this fact is quite standard, see e.g. [1-3, 9, 11]. However, there is one nontrivial moment related to generic properties[1] of systems from these Newhouse regions. One of such properties, called in [3] as the *reversible mixed dynamics*, consists in the fact that, in the case of two-dimensional reversible maps,

- maps having infinitely many coexisting periodic attractors, repellers, saddles and elliptic orbits are generic in these Newhouse regions (in which maps with symmetric homoclinic tangencies are dense), as well as those maps for which the closures of the sets of the orbits of different types have nonempty intersections.

The existence of such *Newhouse regions with reversible mixed dynamics* was proved in [2,3,11] for some cases of one parameter families unfolding generally symmetric couples of heteroclinic and homoclinic tangencies. In [9] this result was proved for C^r-perturbations with $r \leq \infty$ conserving the reversibility. However, this problem, called in [3] as the *Reversible Mixed Dynamics conjecture* (RMD-conjecture), remains to be widely open for the analytical and multidimensional cases and, what is the most important, for one parameter families of reversible two-dimensional maps. In the latter case, the main problem consists in the study of *global symmetry breaking bifurcations*, i.e. such global bifurcations that lead to the birth of a symmetric couple of periodic orbits of type "attractor-repeller" or "saddle($J > 1$)-saddle($J < 1$)".

[1] Recall that a property of systems from an open set D is called generic, if it holds for a residual subset of D, i.e. such subset which can be obtained as result of a countable intersection of open and dense subsets of D.

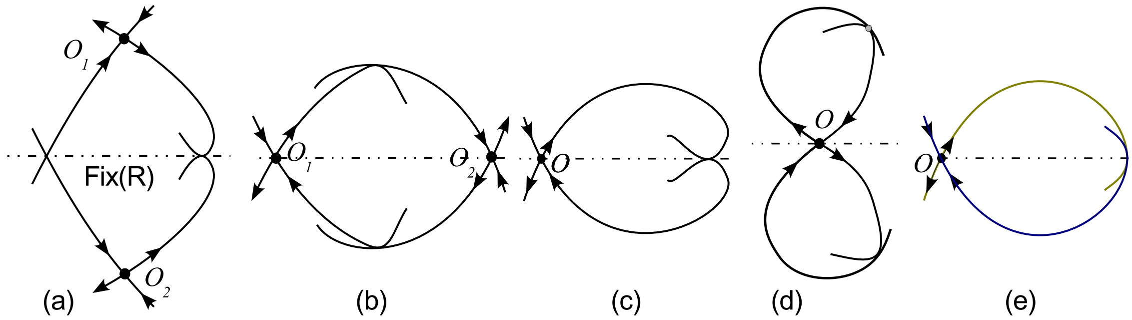

Figure 3: Examples of two-dimensional reversible maps with symmetric homoclinic and heteroclinic tangencies. Maps with symmetric nontransversal heteroclinic cycles are shown in figs (a) and (b): here (a) $O_1 = R(O_1)$ and $J(O_1) = J(O_2)^{-1} < 1$, (b) $J(O_1) = J(O_2) = 1$. Maps with symmetric homoclinic tangencies are shown in figs (c)–(e): here the point O is symmetric in all cases; the homoclinic orbit is symmetric in the cases (c) and (e) where examples with a quadratic and a cubic homoclinic tangencies are sown, resp.; (d) an example of reversible map with a symmetric couple of quadratic homoclinic tangencies to O.

We note that the main local symmetry breaking bifurcations are well-known, see e.g. [10], these are, first of all, pitch-fork bifurcations of various types. Concerning global symmetry breaking bifurcations, they have been studied only for some partial cases of two-dimensional reversible maps. In particular, such bifurcations were investigated for the cases (a), (b) and (d) of Fig. 3 in [2], [3] and [11], respectively. Note that global symmetry breaking bifurcations for the homoclinic cases (c) and (e) of Fig. 3 are still not studied (in the framework of general one parameter unfoldings).

The discovery of the phenomenon of mixed dynamics, and especially reversible mixed dynamics, has the important meaning for the theory of dynamical chaos as whole. The point is that the corresponding type of chaotic orbit behavior can not be associated neither with the dissipative (strange attractor) nor with conservative form of chaos. Therefore, it should be considered as the third independent form of dynamical chaos. Main distinct of (reversible) mixed dynamics consists in the fact that "attractor" and "repeller" intersect here but do not coincide. Indeed, by any definition, "attractor" is a closed invariant set that has to contain all stable periodic points, analogously, "repeller" should contain all completely unstable periodic orbits. Then, the mixed dynamics implies automatically the required intersection.

However, we need to give in this situation more or less adequate definition for "attractor" and "repeller". For the case of two-dimensional reversible maps with dim $Fix(R) = 1$ we can define these closed invariant sets using the notation of ε-trajectories. Recall the corresponding definitions.

Definition 1 *Let $f : M \to M$ be a diffeomorphism defined on some manifold M and let $\rho(x, y)$ be the distance between the points $x, y \in M$. A sequence of points $x_n \in M$ such that*

$$\rho(x_{n+1}, f(x_n)) < \varepsilon, \quad n \in \mathbb{Z}$$

*is called an ε-*orbit *of the diffeomorphism f. If $n \in \{0, 1, 2, ...\}$ we say on an ε^+-*orbit *and if $n \in \{0, -1, -2, ...\}$ on an ε^--*orbit.

Definition 2 *We will call a point y achievable from a point x via ε-orbits (ε-achievable) if for any $\varepsilon > 0$ there exist an ε-orbit of the point x passing through the point y.*

Definition 3 *An* attractor *of a point x is the ω-limit set $\mathcal{A}_x$ of its achievable via ε^+-orbits for any $\varepsilon > 0$. Accordingly, a* repeller *of a point x is the α-limit set $\mathcal{R}_x$ of its achievable via ε^--orbits for any $\varepsilon > 0$. So that attractor $\mathcal{A}_S$ (resp., repeller $\mathcal{R}_S$) of some set S is a union of the corresponding attractors (resp., repellers) of all its points.*

Let $\mathcal{L}_\varepsilon$ be a set of points of all ε-orbits of the point x. Then we can write[2]

$$\mathcal{A}_x = \bigcap_{\varepsilon>0} \overline{\bigcup_{n>0} x_n \in \mathcal{L}_\varepsilon} \text{ and } \mathcal{R}_x = \bigcap_{\varepsilon>0} \overline{\bigcup_{n<0} x_n \in \mathcal{L}_\varepsilon}.$$

Note that, in the reversible case, if a point x belongs to the domain of attraction of some periodic sink p_s, then $\mathcal{A}_x = p_s$ and $\mathcal{R}_x = R(p_s)$. The situation can be more complicated when x is saddle or elliptic periodic point, or homoclinic/heteroclinic point, then an attractor (repeller) such a point can be not trivial.

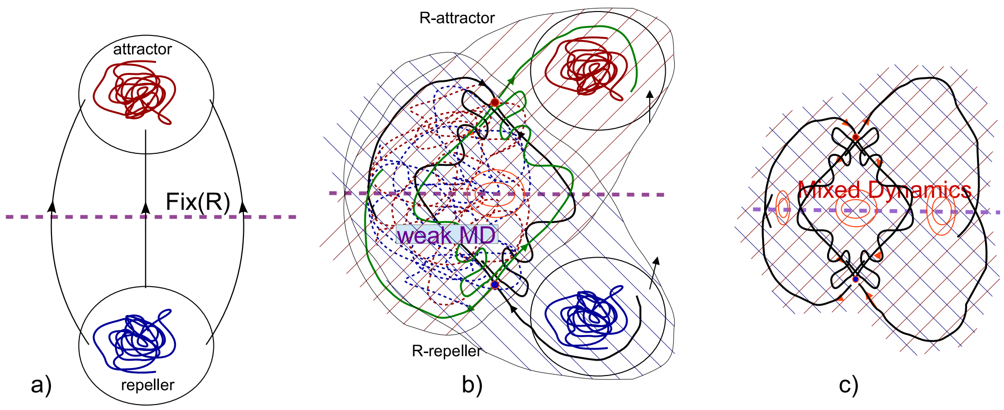

Figure 4: Schematic pictures of various types of chaotic behavior in two-dimensional reversible maps: (a) attractor and repeller are separated, $\mathcal{A} \cap \mathcal{R} = \emptyset$; (b) "the conservative chaos coexists with the dissipative behavior", $\mathcal{A} \cap \mathcal{R} \neq \emptyset$ and there exist an adsorbing domain containing a strange attractor (a symmetric to it strange repeller exists also); (c) reversible mixed dynamics, $\mathcal{A}$ and $\mathcal{R}$ almost coincide.

Definition 4 *Let f be a R-reversible two-dimensional diffeomorphisms and* $\dim Fix(R) = 1$. *The sets $\mathcal{A} = \mathcal{A}_{FixR}$ and $\mathcal{R} = \mathcal{R}_{FixR}$ are called an R-*attractor *and an R-*repeller *of f. The invariant set $\mathcal{RC} = \mathcal{A} \bigcap \mathcal{R}$ is called an R-*core.

[2] If to put $n \in \mathbb{Z}$ in these formulas, then we obtain the definition of the prolongation of the point x. Thus, the attractor (resp., repeller) of the point x is its prolongation by ε^+-orbits (resp., by ε^+-orbits).

Evidently, $R(\mathcal{A}) = \mathcal{R}$. The cases when $\mathcal{RC} = \emptyset$ are well-known, in these cases the attractor and repeller are posed in different parts of the phase space, see e.g. Fig. 4a). If f is an area-preserving and reversible map, then $\mathcal{RC} = \mathcal{A} = \mathcal{R}$. The most interesting cases are those where $\mathcal{RC} \neq \emptyset$ and $\mathcal{A} \neq \mathcal{R}$. The first such case was observed in [15] which was labeled as "the conservative chaos coexists with the dissipative behavior", see also [14]. Now we can say that a kind of mixed dynamics was observed here when the sets $\mathcal{A}$ and $\mathcal{R}$ are essentially different, see e.g. Fig. 4b). Recently it was discovered a new type of mixed dynamics when reversible attractor and repeller almost coincide [12,13].

Schematically this situation can be represented as in Fig. 4c), when the chaotic set becomes bigger comparing with the case of Fig 4b), due to appearance of symmetric homoclinic and heteroclinic orbits of all possible types.

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Homoclinic tangencies in area-preserving and orientation-reversing maps

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We study bifurcations of area-preserving and orientation-reversing maps with quadratic homoclinic tangencies. We study the case when the maps are given on non-orientable two-dimensional surfaces as well as the case of maps with non-orientable saddles. We consider one and two parameter general unfoldings and establish the results related to the emergence of elliptic periodic orbits. This is a joint work with A. Delshams and S. Gonchenko.

On interrelation between dynamics and topology of ambient manifold

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Our lectures are devoted to Morse-Smale systems on closed orientable 3-manifold M^3. The main goal is to present some results on interrelations between topology of M^3 and dynamics of Morse-Smale systems acting on it. We will give scetches of proofs of the statements below and discuss their applications [1].

Let
$$g = \frac{k - \ell + 2}{2},$$
where k is the number of saddles and ℓ is the number of sinks and sources of gradient-like flow (Morse-Smale diffeomorphisms) given on M^3.

Theorem 1 [2] There exists a gradient-like flow (Morse-Smale diffeomorphism) without heteroclinic trajectories (curves) on M^3 if and only if M^3 is the sphere S^3 and $k = \ell - 2$, or M^3 is the connected sum of g copies of $S^2 \times S^1$.

Theorem 2 [3] If M^3 admits gradient-like flow (diffeomorphism with tame frame of one-dimensional separatrices of saddles) then the manifold M^3 admits the Heegaard spliting of genus g.

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On constructing simple examples of three-dimensional flows with two heteroclinic cycles

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It is well known that systems with heteroclinic cycles are not structurally stable among the generic dynamical systems. However, they could be structurally stable if they have additional properties. For example, equivariant systems [1] or generalized Volterra-Lotka systems [2] can possess structurally stable heteroclinic cycles. Studying the properties of heteroclinic networks is a topic of interest because these structures appear in models of different phenomena of natural sciences including physics [1], chemistry, mathematical biology [2] and neuroscience [3]. Although there are many papers devoted to implementing heteroclinic networks with given topology (e.g., [4]) it seems that low-dimensional systems that possess two heteroclinic cycles from different networks are less studied subject. In this work we suggest a simple method for constructing $Z_3 \times (Z_2)^3$-equivariant systems of ODEs in R^3 (i.e., systems, whose trajectories are invariant under the action of this group on R^3) that possess two heteroclinic cycles. We assume the action of $Z_3 \times (Z_2)^3$ on R^3 to be generated by cyclic permutations of coordinate axes and reflection with respect to one of the coordinate planes. Also we present a sketch of global dynamics analysis for "the simplest" example.

"The simplest" example comes from the question what global dynamics can have three-dimensional system with two heteroclinic cycles. The equilibrium at origin is assumed to be a sink, one of the heteroclinic cycles is attractor and another one is repeller. Also we assume that there is no other equilibria or heteroclinic cycles except mentioned above. If we consider only polynomial vector fields, the minimal (in the sense of degree) example can be constructed as follows. Choose functions

$$\mathcal{R}(u,v) = -\mu(u^2+v^2-1)(u^2+v^2-R^2),\ \Phi(u,v) = (x^2+\varepsilon y^2)(x^2+\varepsilon y^2 - s),$$

$$M(u,v) = \mathcal{R}(u,v) - v^2\Phi(u,v),\ N(u,v) = \mathcal{R}(u,v) + u^2\Phi(u,v),$$

$$V(u,v,w) = \mu\zeta vw,\ S(u,v,w) = M(u,v) + N(w,u) - M(u,0) + vw\cdot V(u,v,w).$$

The full three-dimensional system has form

$$\dot{x} = x\cdot S(x,y,z),\ \dot{y} = y\cdot S(y,z,x),\ \dot{z} = z\cdot S(z,x,y).$$

Appropriately choosing parameters and taking into account the requirements that were mentioned above, we observe that global dynamics could be described the following way (also see Fig. 1). An invariant bottleneck-shaped surface, which rests on inner heteroclinic cycle, separates two basins of attraction and divides R^3_+ into two connected components. Trajectories that start in the same connected component as the points on diagonal $x = y = z$ are attracted by sink at the origin; trajectories that start outside are attracted by the outer heteroclinic cycle. Since inner heteroclinic cycle is repeller, there are no other equilibria except those on coordinate planes and trajectories on invariant surface don't leave the spherical segment between

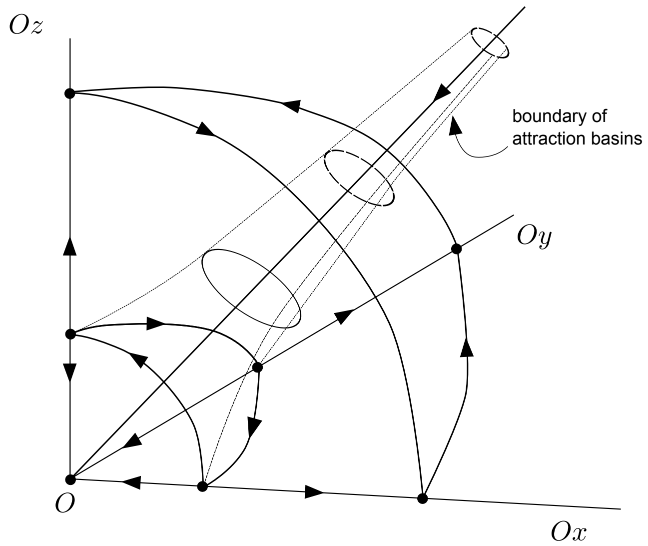

Figure 5: Sketch of global dynamics

two heteroclinic cycles, restriction of dynamics onto the invariant surface should have at least one limit cycle.

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Cascades of bifurcations in two-parameter ecological system

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The model Hastings - Powell's ecological system of the "prey-predator-top-predator" describes a system of differential equations with parameters:

$$\zeta\dot{x} = x(1-x) - \frac{\alpha_1 xy}{1+\beta_1 x}, \dot{y} = y(\frac{\alpha_1 xy}{1+\beta_1 x} - \delta_1) - \frac{\alpha_2 yz}{1+\beta_2 y}, \dot{z} = \varepsilon z(\frac{\alpha_2 y}{1+\beta_2 y} - \delta_2). \qquad (8)$$

As a bifurcation parameters considered β_1 and δ_2 and the parameters ζ, ε, α_1, α_1, β_2, δ_1 are fixed. For a singular point $O(x^*, y^*, z^*)$ which is in the region of positive x, y, z built a partition of the plane parameters β_1 and δ_2 into the regions according to the type of the coarse singular point of the linearized system. When crossing the boundary of a saddle-focus with positive real parts of a pair of complex conjugate roots going Andronov-Hopf bifurcation of birth of a stable limit cycle, followed by a cascade of period-doubling bifurcations cycle and subharmonic cascade Sharkovskii ending cycle period of the birth of three. A further change in the parameters appear in the system cycles homoclinic bifurcation cascade leading to the formation of a strange attractor. With transforms laid computing systems and evidence shows the existence of a homoclinic orbit of a saddle-focus, the destruction of which is the principal homoclinic bifurcation cascade and defined range of parameters in which it exists. Bifurcation diagrams, graphs Lyapunov exponents of saddle graphics, fractal dimension of the strange attractor. Work is executed with the use of analytical and numerical calculations Maple.

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Mixed-mode oscillations and twin canards

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A mixed-mode oscillation (MMO) is a complex waveform with a pattern of alternating small- and large-amplitude oscillations. MMOs have been observed experimentally in many physical and biological applications, most notably in chemical reactions. We are interested in MMOs that appear in an autocatalytic system with one fast and two slow variables. The mathematical analysis of MMOs is very geometric in nature and based on singular limits of the time-scale ratios. Near the singular limit one finds so-called slow manifolds that guide the dynamics on the slow time scale. In systems with one fast and two slow variables, slow manifolds are surfaces that can be either attracting or repelling. Transversal intersections between attracting and repelling slow manifolds are called canard orbits, and they organise the observed patterns of MMOs. Our aim is to study a parameter regime where the time-scale ratio is relatively large. Here, the structure of MMO patterns is richer and different from that predicted by the theory for small time-scale ratios. To study the underlying geometry, we use advanced numerical methods based on the continuation of orbit segments defined by a suitable boundary value setup. We find that canard orbits appear in pairs, which we call twin canards. Twin canards bound regions of different numbers of small-amplitude oscillations. We also perform parameter continuations of canard orbits to identify parameter regimes of different small-amplitude oscillations.

From wild Lorenz-like to wild Rovella-like dynamics

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We consider a two-dimensional noninvertible map that was introduced by Bamón, Kiwi and Rivera in 2006 as a model of wild Lorenz-like chaos. The map acts on the plane by opening up the critical point to a disk and wrapping the plane twice around it; points inside the disk have no preimage. The bounding critical circle and its images, together with the critical point and its preimages, form the critical set. This set interacts with a saddle fixed point and its stable and unstable sets. Advanced numerical techniques enable us to study how the stable and unstable sets change as a parameter is varied along a path towards the wild chaotic regime. We find four types of bifurcations: the stable and unstable sets interact with each other in homoclinic tangencies (which also occur in invertible maps), and they interact with the critical set in three types of tangency bifurcations specific to this type of noninvertible map; all tangency bifurcations cause changes to the topology of these global invariant sets. Overall, a consistent sequence of all four bifurcations emerges, which we present as a first attempt towards explaining the geometric nature of wild chaos. Using two-parameter bifurcation diagrams, we show that essentially the same sequences of bifurcations occur along different paths towards the wild chaotic regime, and we use this information to obtain an indication of the size of the parameter region where wild Lorenz-like chaos is conjectured to exist. We further continue these bifurcations into a regime where the dynamics change from Lorenz-like to Rovella-like, that is, where the equilibrium contained in the attractor becomes contracting. We find numerical evidence for the existence of wild Rovella-like attractors, wild Rovella-like saddles and regions of multistability, where a Rovella-like attractor coexists with two fixed-point attractors.

On synchronization of hyperbolic chaotic generators and based on it communication schemes

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In the last decades the phenomenon of chaotic synchronization attracts much attention, especially from the point of view of technical applications. A special type of chaotic behavior is a hyperbolic chaos, possessing the property of robustness. A number of physically realizable models of generators with such chaotic behavior have been developed just recently [S.P. Kuznetsov. Hyperbolic chaos: A physicist's view. Springer. 2011]. The subject of this work is investigation of some of the mechanisms of transition to synchronous behavior and characteristics of dynamical and statistical properties of these mode for unidirectionally coupled hyperbolic chaotic generators, namely, one example of such a generator, which is associated to the Smale-Williams attractor. Special attention is paid to the description of robustness and hyperbolicity of the dynamics of such "master-slave" system on the road to synchrony, including the case of nonidentical by parameters subsystems. Results of numerical simulation as well as radio-electronic experiment are provided.

Moreover, in this work several schemes of chaotic communication, which functional elements (transmitter and receiver) are the generators of robust hyperbolic chaos, are proposed. These generators are associated to the dynamics of: 1) expanding circle map (Bernoulli map); 2) conservative "Arnold cat" map; 3) hyperchaotic map. To implement the confidentiality of communication the approach of nonlinear mixing of information signal to the chaotic signal of transmitter is applied. Detection of the desired information is produced in receiver by its synchronization with the transmitter. Using of the generators of robust chaos insensitive to small disturbances and perturbations is appeared to be a good choice for communication schemes.

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Dynamical Self-Assembly Processes

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A class of dynamical models describing general features of chemical, biological and social systems is treated. In the frame of kinetic approach, such a system is characterized with concentrations of components. The concentrations satisfy a multidimensional systems of first order non-linear ordinary-derivative equations.

In a general case, the equations can be analyzed using multidimensional integrated varieties and, accordngly, multidimensional attractors. Sufficient conditions for the equuilibrium uniqueness and stability are found. If an interaction between subsystems has a hypercycle, relevant concentrations auto-oscillate. Under certain simple restrictions on parameters, the systems of equations may be integrable in quadratures.

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On Chaotic Dynamics in the Suslov Problem

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We note that the recently increased interest in problems of nonholonomic mechanics is due, first, to the development of new methods of numerical analysis and, second, to a great importance of such problems for applications in mobile robotics.

For more on the stages of historical development of nonholonomic mechanics see, for example, [1]. Here we would only like to note that systems of nonholonomic mechanics exhibit a greater variety of dynamical behaviors than Hamiltonian systems [2]. This variety of behaviors stems, on the one hand, from the presence (or absence) of various *tensor invariants* in the system (such as first integrals, invariant measure, and Poisson structure) and, on the other hand, from the *reversibility* of the system and the number and type of *involutions*.

As an example of the variety of behaviors in nonholonomic systems we consider two particularly interesting problems which are fundamentally different from a dynamical point of view: the Chaplygin ball rolling problem [3] and the problem of motion of a rattleback [4,5]. The former is described by an integrable system, while the latter is nonintegrable and, moreover, exhibits strange attractors [6,7].

In this abstract we consider the nonholonomic Suslov problem describing the motion of a heavy rigid body with a fixed point, subject to a nonholonomic constraint, which is expressed by the condition that the projection of angular velocity onto the body-fixed axis is equal to zero. For convenience, we call such a body the *Suslov top*.

We show that in the general case (in a gravitational field) the Suslov problem does not possess an invariant measure. In this case, in the phase space of the system (depending on parameters) there can exist different limiting regimes ranging from regular attractors (which are not always expressed in terms of quadratures) to strange attractors.

In Section 1 we present equations of motion for the Suslov top and find first integrals. In Section 2 we describe the procedure of constructing a Poincaré map for numerical analysis of the system and present a complete list of involutions in the system.

Section 3 presents results on chaotic dynamics, which have been obtained by analyzing the charts of Lyapunov exponents, and introduces numerical criteria for the classification of chaotic dynamics. This criterion made it possible to show that in addition to simple attractors and repellers (by virtue of reversibility) and complex strange attractors, the system has zones where conservative and dissipative behaviors alternate with each other. Such behavior, called *mixed dynamics*, was theoretically proven in [8,9] for general reversible systems, and was previously detected in other nonholonomic systems [7, 10].

Highly efficient numerical experiments were conducted using the software package "Computer Dynamics: Chaos" by means of a computational cluster of the laboratory LATNA of the National Research University Higher School of Economics. This abstract is the part of our joint work with I.A.Bizyaev and A.V.Borisov [12].

1. Equation of motion. Let us consider the motion of a heavy rigid body with a fixed point in the presence of the nonholonomic constraint

$$(\boldsymbol{\omega}, \boldsymbol{e}) = 0, \tag{9}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body and $\boldsymbol{e}$ is the unit vector fixed in the body.

The constraint (9) was introduced by G. K. Suslov in p. 593 [13]. The realization of the constraint (9) by means of wheels with sharp edges rolling over a fixed sphere was proposed by V. Vagner [14] (see Fig. 6). The sharp edges of the wheels prevent the wheels from sliding in the direction perpendicular to their plane.

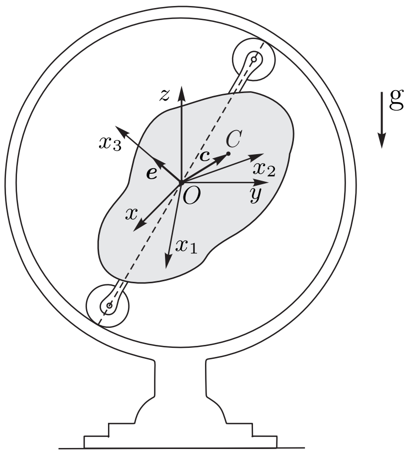


Figure 6: Realization of the Suslov problem.

Choose two coordinate systems:

— *an inertial* (fixed) coordinate system $Oxyz$;

— *a noninertial* (moving) coordinate system $Ox_1x_2x_3$ rigidly attached to the rigid body in such a way that $Ox_3 \| \boldsymbol{e}$ and the axes Ox_1 and Ox_2 are directed so that one of the components of the tensor of inertia of the body vanishes: $I_{12} = 0$.

To parameterize the configuration space, we choose a matrix of the direction cosines $\mathbf{Q} \in SO(3)$ the columns of which contain the unit vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ of the fixed axes Ox, Oy and Oz projected onto the axes of the moving coordinate system $Ox_1x_2x_3$

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in SO(3).$$

In the moving coordinate system $Ox_1x_2x_3$ the constraint equation (9) and the tensor of inertia $\mathbf{I}$ of the rigid body have the form

$$\begin{gathered} \omega_3 = 0, \\ \mathbf{I} = \begin{pmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}. \end{gathered} \tag{10}$$

Let $\boldsymbol{c} = (c_1, c_2, c_3)$ be the vector of displacement of the center of mass of the body relative to the fixed point O and assume that the entire system is in a gravitational field with the potential

$$U = (\boldsymbol{b}, \boldsymbol{\gamma}), \quad \boldsymbol{b} = -m\mathrm{g}\boldsymbol{c},$$

where m is the mass of the rigid body and g is the free fall acceleration.

The equations of motion for $\boldsymbol{\omega}$ in the moving coordinate system $Ox_1x_2x_3$ have the form

$$\begin{gathered} \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \lambda \boldsymbol{e} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \\ \lambda = -\frac{\left(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \mathbf{I}^{-1}\boldsymbol{e}\right)}{(\boldsymbol{e}, \mathbf{I}^{-1}\boldsymbol{e})}, \end{gathered} \tag{11}$$

where $\boldsymbol{e} = (0, 0, 1)$.

Adding to the system (11) the kinematic Poisson equations governing the evolution of the unit vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$:

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{12}$$

we obtain a complete system governing the motion of the rigid body.

In Eqs. (11) and (12), a closed system for the variables $(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3)$ decouples. In view of the constraint (10) this system can be represented as

$$\begin{aligned} & I_{11}\dot{\omega}_1 = -\omega_2(I_{13}\omega_1 + I_{23}\omega_2) + b_3\gamma_2 - b_2\gamma_3, \\ & I_{22}\dot{\omega}_2 = \omega_1(I_{13}\omega_1 + I_{23}\omega_2) + b_1\gamma_3 - b_3\gamma_1, \\ & \dot{\gamma}_1 = -\gamma_3\omega_2, \\ & \dot{\gamma}_2 = \gamma_3\omega_1, \\ & \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1. \end{aligned} \tag{13}$$

The system (13) possesses an energy integral and a geometric integral:

$$E = \frac{1}{2}(I_{11}\omega_1^2 + I_{22}\omega_2^2) + (\boldsymbol{b}, \boldsymbol{\gamma}), \quad F_1 = \boldsymbol{\gamma}^2 = 1. \tag{14}$$

Thus, on the fixed level set of the energy integral $E = h$ and $F_1 = 1$ the system (13) defines the flow on the three-dimensional manifold $\mathcal{M}_h^3$:

$$\mathcal{M}_h^3 = \{(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3), \mid E = h, \ F_1 = 1\},$$

for it to be integrable by the Euler–Jacobi theorem, an additional first integral F_2 and an invariant measure are necessary.

2. The Poincaré map and involutions. In the general case, Eqs. (13) define the flow $\mathcal{F}$ on the three-dimensional manifold

$$\mathcal{M}_h^3 = \{(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3) \mid E = h,\ F_1 = 1\}.$$

To parameterize this flow, we shall use the variables ω_1, γ_1 and γ_2 by expressing ω_2 and γ_3 in terms of the integrals (14):

$$(\dot{\omega}_1, \dot{\gamma}_1, \dot{\gamma}_2) = \mathcal{F}(\omega_1, \gamma_1, \gamma_2)^3.$$

Choosing the plane γ_1 = const as the secant of the three-dimensional flow $\mathcal{F}$, we obtain a two-dimensional Poincaré map[4]

$$(\bar{\gamma}_2, \bar{\omega}_1) = \mathcal{P}(\gamma_2, \omega_1). \tag{15}$$

Since the variables γ_3 and ω_2 are defined in terms of the integrals repeatedly, the resulting map is many-leaved (the choice of the signs γ_3 and ω_2 defines a specific leaf). For definiteness, in what follows we shall choose a leaf corresponding to positive values of the variable ω_2.

On the constructed two-dimensional Poincaré map $\mathcal{P}$, the fixed points correspond to periodic orbits (cycles) in the initial system (13).

Reversibility and involutions

The studies [15, 16] show that the presence of reversibility and the number of involutions in the system considerably influence the type and complexity of the dynamics of nonholonomic systems. The papers [15] are concerned with the motion of rigid bodies of different forms moving on the surface without slipping and spinning. In these papers it is shown that, depending on the geometrical and dynamical properties of the body, the system can have different numbers of involutions, which ultimately determines the type of chaotic dynamics in the system. The results of investigation of the Chaplygin top (a dynamically asymmetric ball with a displaced center of mass) are presented in [16]. In the case of an arbitrary displacement of the center of mass of this top the system is reversible under the only involution, and the top itself can execute a reversal (like a rattleback). Moreover, a strange attractor of figure-of-eight type was found in this case. The papers [6,7] are concerned with the motion of a rattleback. In these papers it is also noted that the rattleback dynamics is also connected with involutions.

We now turn to analysis of reversibility in the system (13). In the general case (with any parameters) the system is reversible only under one involution

$$R_0 : \omega_1 \to -\omega_1, \quad \omega_2 \to -\omega_2, \quad t \to -t. \tag{16}$$

Remark 1 *We recall that a system is said to be reversible under involution R if this system is invariant under R and the time reversal $t \to -t$, and the transformation $R \circ R$ is an identity transformation.*

Due to this involution the phase portrait of the system (13) possesses the following properties:

- For each trajectory $\mathcal{F}(\gamma_1, \gamma_2, \omega_1)$ there exists a symmetric (relative to $Fix(R_0) = \{\omega_1 = \omega_2 = 0\}$) trajectory $\mathcal{F}^{-1}(R_0(\gamma_1, \gamma_2, \omega_1))$ which is in involution with the initial one.

[3] In this case ω_2 and γ_3 are easily expressed in terms of the energy and geometric integrals. Since the variables ω_1 and ω_2 appear in Eqs. (13) equivalently, we can write in a similar way the three-dimensional flow in the variables $(\omega_2, \gamma_1, \gamma_2)$.

[4] For more on the procedure of constructing Poincaré maps for various problems of rigid body dynamics.

- If the set A is an attractor, then the set $R_0(A)$ is attracting for the flow in reverse time $\mathcal{F}^{-1}$, i.e., it is a repeller.

In paper [12] it is shown that the properties described above give rise to a reversal in the system (13) and that this reversal can be of the same type as for rattlebacks.

Consider the Suslov top whose center of mass is displaced only along the axis Ox_3, i.e., $\boldsymbol{b} = (0, 0, b_3)$. In this case, depending on the tensor of inertia $\mathbf{I}$, additional involutions may appear in the system (13). A complete list of these involutions is presented in the table below:

Table 1: Additional involutions of the system (13) for $\boldsymbol{b} = (0, 0, b_3)$.

	$I_{13} = I_{23} = 0$	$I_{13} = 0,\ I_{23} \neq 0$	$I_{13} \neq 0,\ I_{23} = 0$	$I_{13} \neq 0,\ I_{23} \neq 0$
R_1: $\omega_1 \to -\omega_1$, $\gamma_1 \to -\gamma_1$, $t \to -t$	$+$	$+$	$-$	$-$
R_2: $\omega_2 \to -\omega_2$, $\gamma_2 \to -\gamma_2$, $t \to -t$	$+$	$-$	$+$	$-$

In order to carry over the above-mentioned involutions to the Poincaré map (15), we have to define as a secant a manifold that is invariant under an involution. Therefore, the most suitable secant for the system (13) is the hyperplane $\gamma_1 = 0$. We note that on the Poincaré map (15) we work with a leaf that corresponds to a specific positive value of the variable ω_2, and hence some involutions (for example, R_0 and R_2) cannot be carried over.

Thus, under the choice of the secant $\gamma_1 = 0$ and the additional condition $I_{13} = 0$ the constructed Poincaré map (15) can possess the only involution

$$r_1 : \omega_1 \to -\omega_1, \tag{17}$$

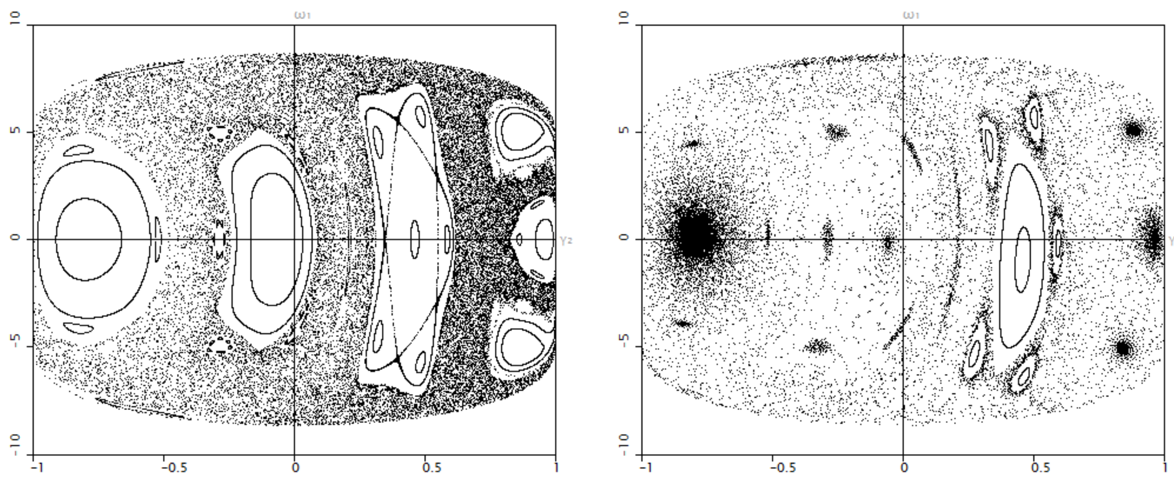
whose set of fixed points forms the straight line

$$Fix(r_1) = \{\omega_1 = 0\}.$$

In Section 3.1 we will use the straight line $Fix(r_1)$ for the investigation and classification of chaotic dynamics. For convenience, we introduce the following definitions. We use the term *reversible attractor* for a limiting set formed by iterations (on the Poincaré map) of the line $\omega_1 = 0$ in direct time and the term *reversible repeller* for a limiting set consisting of iterations of this line in reverse time. By the Poincaré reversibility theorem [17], in the case of existence of a smooth invariant measure in the system, the reversible attractor and the reversible repeller are undistinguishable. Otherwise these sets are distinguishable, but symmetric to each other relative to $Fix(r_1) = \{\omega_1 = 0\}$. In Section 3.1 we shall use the degree of such distinguishability for the classification of chaotic regimes.

Figure 7 shows phase portraits of the system (13) with different parameters. For the Suslov top for which $I_{13} = 0$, the Poincaré map is involutive relative to the straight line $\omega_1 = 0$ (see Fig. 7a).

In the general case, the Poincaré map (15) admits no involutions and exhibits visible crowdings of points (practically black regions) corresponding to simple attractors, which are fixed and periodic points (see Fig. 7b). As is well known, the existence of any attractors is an obstacle to the existence of a smooth invariant measure. Moreover, the presence of a chaotic layer is indicative of the absence of the additional first integral F_2.



a) $I_{13} = 0$ b) $I_{13} = 0.1$

Figure 7: Phase portrait on the Poincaré map (15). All parameters, except for I_{13}, are chosen as follows: $E = 50, I_1 = 4, I_2 = 3, I_{23} = 0, \boldsymbol{b} = (0, 0, 100)$. a) Due to the presence of the involution r_1 the phase portrait on the Poincaré map looks symmetric relative to the horizontal axis. b) When $I_{13} \neq 0$, the involution r_1 disappears, and the phase portrait on the Poincaré map becomes nonsymmetric. Moreover, on the Poincaré map one can see crowdings of trajectories near asymptotically stable points of different periods.

Chaotic dynamics. In this section we present numerical results on the chaotic dynamics in the nonholonomic Suslov model and show that the system under consideration exhibits complex chaotic behavior whose type essentially depends on the system parameters and hence on the number of involutions.

We shall classify various limiting (including chaotic) regimes by analyzing the charts of Lyapunov exponents on the parameter plane (I_{23}, E) by fixing the other system parameters.

We describe the scheme of constructing the charts of Lyapunov exponents[5]. We divide the parameter plane (I_{23}, E) into 400×400 nodes and start from each node a trajectory on the Poincaré map (15) with some initial conditions (γ_2, ω_1). To preclude a transient process, the system was integrated for $T = 4 \cdot 10^4$ time units, and then the Lyapunov exponents were estimated on the interval $T = 10^4$ by the Benettin method [18]. Depending on the values of λ_1, λ_2 and λ_3, we color the corresponding node in the chart in a particular color.

We note that the system (13) possesses three essential (nonzero) exponents $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0$. To exclude from consideration two zero exponents, which correspond to the integrals (14), we apply the procedure of normalization of phase variables to the level sets of the integrals [11].

Further we shall consider 2 fundamentally different cases:

- $I_{13} = 0$ — the system (13) is reversible under two involutions: R_0 and R_1;
- $I_{13} \neq 0$ — the system (13) is reversible under the only involution R_0.

Chaotic Dynamics for $I_{13} = 0$ In the case considered, the Poincaré map (15) is symmetric relative to the straight line $Fix(R_1) = \{\omega_1 = 0\}$. To construct a chart of Lyapunov exponents,

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[5] For more on the methods of constructing the charts of Lyapunov exponents, see, for example, in [11, 16], where such charts are constructed for nonholonomic systems describing the rolling motion of a rattleback and the Chaplygin top, respectively. However, in this paper we have slightly modified the procedure of classification of chaotic regimes.

we fix the system parameters by the following values:

$$I_{13} = 0, \quad I_1 = 3, I_2 = 4, b_1 = 0, b_2 = 0, b_3 = 100, \tag{18}$$

and as the initial point for each node of the chart we use a point with coordinates $(\gamma_2, \omega_1) = (0.9, 0.1)$. We give some explanations on the regimes depicted in the chart (see Fig. 8). Among the regular regimes ($\lambda_1 \leqslant 0$) in the chart of Lyapunov exponents we distinguish:

- $\lambda_1 < 0$ — equilibrium;
- $\lambda_1 = 0 \quad \lambda_1 + \lambda_2 + \lambda_3 < 0$ — cycle (a periodic or fixed point on the map (15));
- $\lambda_1 = 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0$ — elliptic orbit (invariant curves around an elliptic point or the elliptic point itself).

In addition to regular limiting regimes in the chart of Lyapunov exponents, we shall also classify various chaotic ($\lambda_1 > 0$) regimes. We note that the case $\lambda_1 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$ corresponds to conservative chaos (see Fig. 8 on the left), in which the phase volume (invariant measure) is conserved. In this case the reversible attractor and the reversible repeller are indistinguishable.

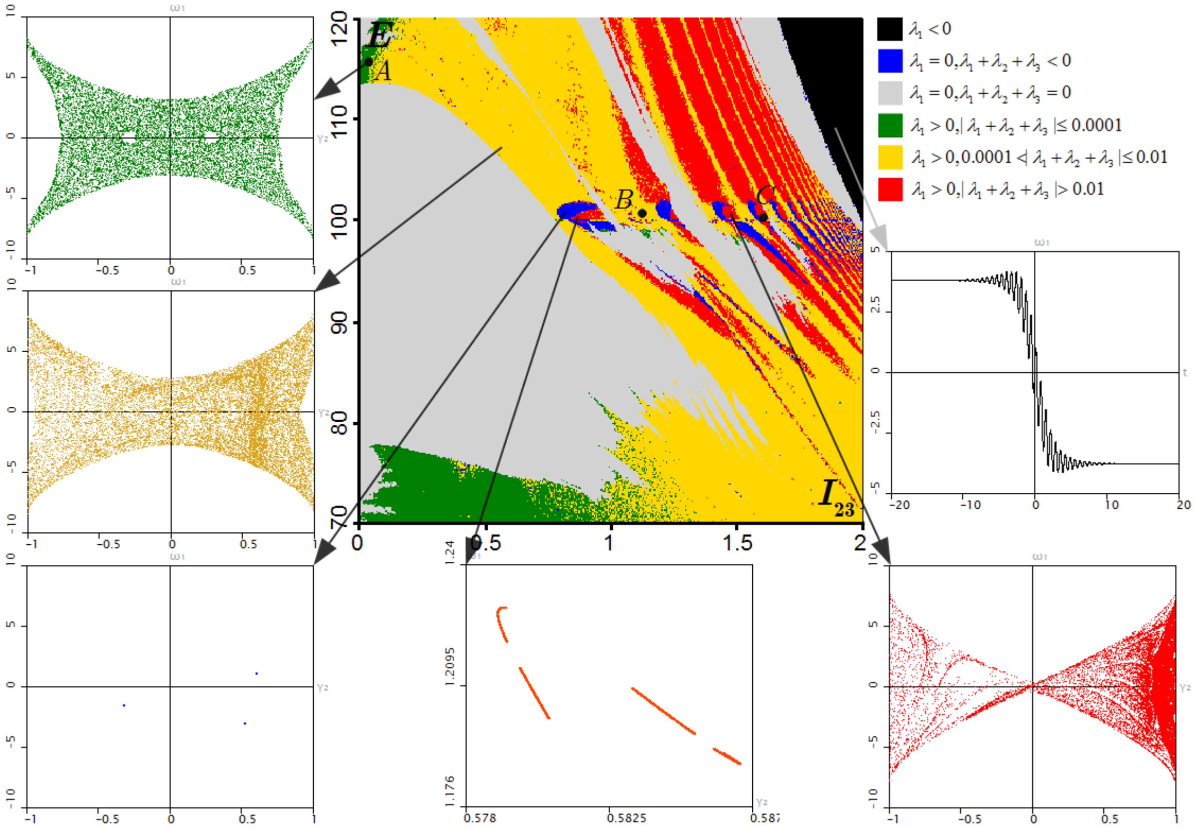
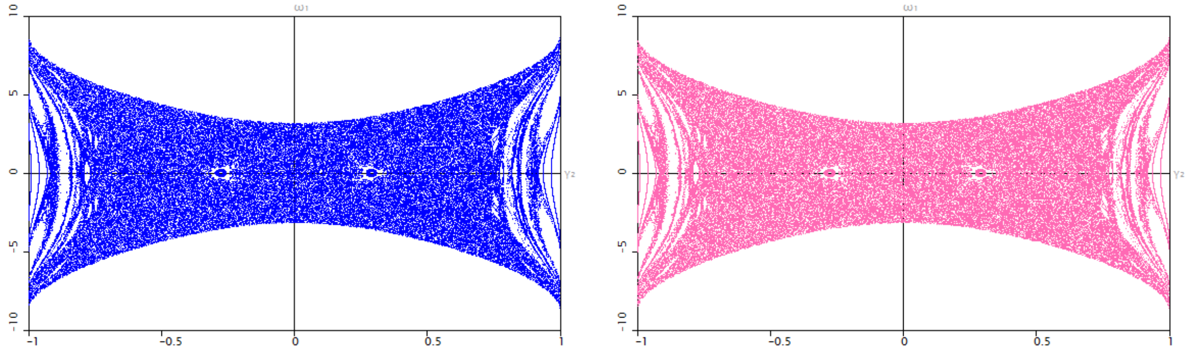


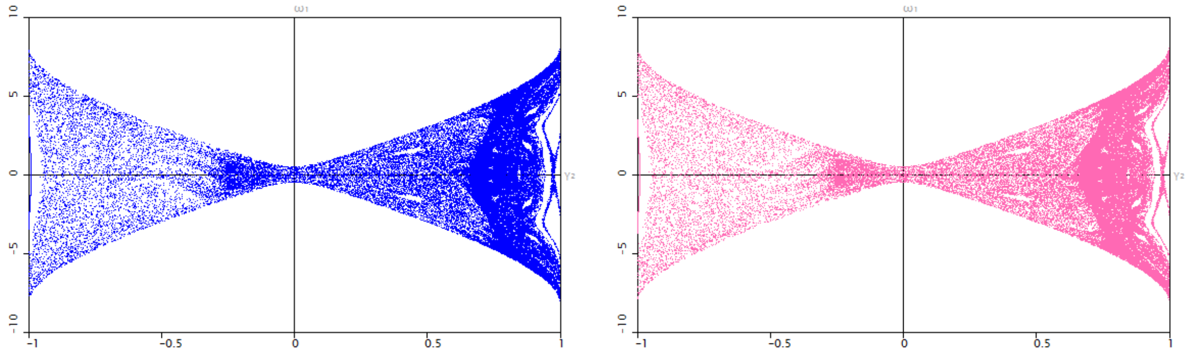
Figure 8: Chart of Lyapunov exponents and the typical phase portraits for different values of the parameters I_{23} and E. The other parameters have been chosen according to (18).

If the system (13) does not possess a smooth invariant measure, then the reversible attractor and the reversible repeller become distinguishable. We have noticed that the degree of such distinguishability can be characterized quite well by the value of the sum of Lyapunov exponents $\lambda_1 + \lambda_2 + \lambda_3$. Depending on this value, we classify the chaotic regimes as follows:



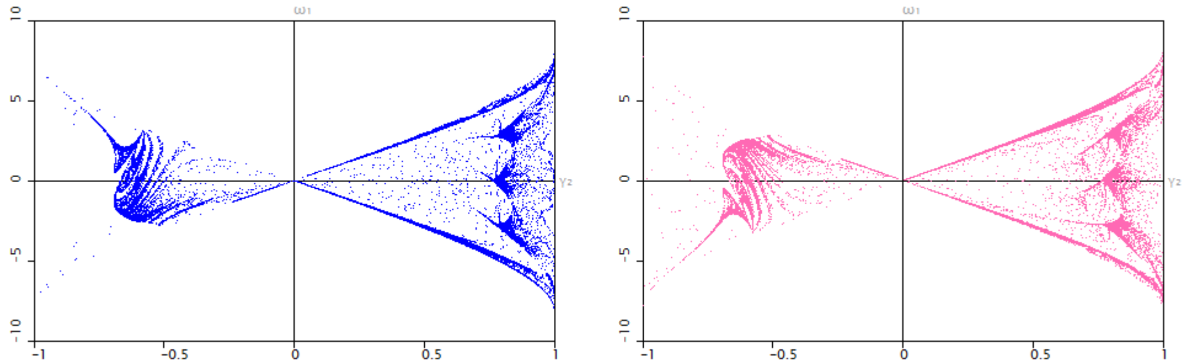
a) b)

Figure 9: Phase portraits on the Poincaré map for the parameters (18), $I_{23} = 0.015, E = 115.125$ (point A in the chart of Lyapunov exponents (see Fig. 8)). (a) the reversible attractor and (b) the reversible repeller are practically indistinguishable.



a) b)

Figure 10: Phase portraits on the Poincaré map for the parameters (18), $I_{23} = 1.5, E = 100$ (point B in the chart of Lyapunov exponents (see Fig. 8)). The reversible attractor and the reversible repeller have a large common part, but are distinguishable nevertheless.



a) strange attractor b) strange repeller

Figure 11: The strange attractor and the strange repeller on the Poincaré map (15) for $I_{23} = 1.5895$ and $E = 100$.

- $\lambda_1 > 0$, $|\lambda_1 + \lambda_2 + \lambda_3| \leqslant 0.0001$. The case considered is close to a conservative one. The reversible attractor and the reversible repeller are practically indistinguishable (see Fig. 9).
- $\lambda_1 > 0$, $0.0001 < |\lambda_1 + \lambda_2 + \lambda_3| \leqslant 0.01$. In this case the reversible attractor and the reversible repeller, although they have a large common part, are distinguishable nevertheless (see Fig. 10).
- $\lambda_1 > 0$, $0.01 < |\lambda_1+\lambda_2+\lambda_3|$. The reversible attractor strongly differs from the reversible repeller, although it is symmetric to it (see Fig. 11). In this case, chaotic dynamics are associated with strange attractors.

  We note that for $0 < |\lambda_1 + \lambda_2 + \lambda_3| < 0.01$ the distinguishability of the reversible attractor and the reversible repeller is connected with the existence (inside the chaotic sea) of stable and completely unstable periodic orbits of different periods that are not separable from each other. Thus, the points of the reversible attractor accumulate near stable periodic points, and the points of the reversible repeller accumulate near completely unstable points that are symmetric to them. Such a structure of a chaotic set is typical of reversible two-dimensional maps without a smooth invariant measure and is called mixed dynamics (see papers [8,9] for more details on this phenomenon and papers [7,10] for examples of mixed dynamics in other nonholonomic reversible systems).

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Modeling of the employed population number nonlinear dynamics: the agent-based approach

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By now, oscillatory and complex modes of dynamics for social and economic systems are still insufficiently studied, and their substantial interpretations are debatable. Some new phenomenological principles in social-economic time series have come to light, and one of them represents structural fluctuations in the number of employed population belonging to different age groups. Modeling of these fluctuations shows that arising dynamic modes of the employed population number depend on such factors as interaction (symbiosis) or competition between the people belonging to different age groups (Khavinson, Kulakov, 2014).

At this, it still remains unclear the role of individual or group strategies in the observed fluctuations of employment. According to the neoclassical economic theory, an economic agent (a person placed in a job), should operate on a labour market in such a way that his activity would maximize his profit. Later, it was added some assumptions to this economic theory that limited rationality of the agent allowing him satisfy his social needs (Simon, 1957; Chernavskii et al, 2011; Chena, Lib, 2012; Khavinson, 2015). It is convenient to use the agent-based approach, realized in the form of a simple model, in order to investigate various employment strategies influencing the number dynamics of employed people (Kolobov, Frisman, 2013; Kolobov, 2014).

In the authorsT' model of dynamics it is considered six age groups of people (agents), employed in three conditional branches of economy. Every branch is estimated on a three-point scale, according to the salary rate, prestige and working conditions. Estimations imply that each branch would lead only at one indicator. To choose the branch, it has been considered 6 strategies, subdivided into pure and combined. The pure strategies mean that an employed person strives to maximize only one of the indicators: salary, prestige or working conditions. The combined strategies reveal a desire of the agent to choose the branch which at most satisfies two criteria: salary and prestige, salary and working conditions, or prestige and working conditions. It is also supposed that every age group is characterized by one and the same strategy of behaviour. Moving of workers between the branches depends on the choice and realization of a concrete strategy. Entry conditions in the model correspond to a uniform distribution of employed people of different age over the branches. The conducted numerical experiments show that a combination of various strategies can lead to a non-uniform distribution of workers, considering their age and conditional branches. The received results underline the importance of studying social behaviour of agents, as this factor has an influence on general dynamics and the employed population distribution.

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Fractal topological structure of the Universe and dark matter problem

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The most important problem of the modern astrophysics is the nature of dark matter. The increasing number of observational facts indicate that dark matter cannot be attributed to weakly interacting particles. It turns out that a satisfactory answer suggests lattice quantum gravity [1]. It was shown there that spacetime foam which exists at sub-Planckian scales possesses fractal properties, while recent numerical simulations [2] demonstrated that the spectral dimension in 4d quantum gravity runs from a value of D= 3/2 at short distance to D = 4 at large distance scales. This indicates the existence of a very complex topological structure of spacetime at very small scales. The inflationary stage in the past has enormously stretched all scales and the fractal quantum spacetime foam structure has been tempered. Such a structure represents a homogeneous and isotropic space filled with a gas of wormholes which represents the initial conditions for the standard cosmological model.

The spacetime fractal structure is described as follows. Consider any point x in space, fix a geodesic distance R, and consider the value of volume $\Omega(R)$ (i.e., it does not depends on the position of the starting point x). Then the scaling $\Omega(R) \propto R^{D_H}$ defines the Hausdorff dimension of space D_H. By the construction it is clear that in 4d gravity $D_H \leq 4$. Indeed, the starting point x together with the omnidirectional jet of geodesic lines define the extrapolating reference system (exactly as we use in astrophysics) for which the coordinate volume scales with $D = 4$. The same behavior works for simple topology spaces. However, in the presence of wormholes for sufficiently big distances R some of geodesics return and start to cover the same (already covered) physical region $\Omega(R)$. Therefore, for sufficiently complex topologies the dimension is always $D_H < 4$.

On the sky we see exactly such a picture. Indeed, the light is too scattered upon propagating through the wormhole throats and it forms the diffused background [3]. Therefore, galaxies are good tracers for the actual (physical) volume which exactly demonstrate the fractal behavior $N(R) \propto R^D$ with $D \approx 2$ up to distances $R \sim 200Mpc$ [4]. In particular, this reflects the long standing puzzle of missing baryons. The nontrivial complex topology leads to the modification of Newton's law [5] which is interpreted as the dark matter phenomenon and which perfectly fits the observations [6].

In asymptoticaly flat spaces stable wormholes are known to require the presence of some amount of exotic matter. However the problem of the stability is solved if we realize the gas of wormholes as a space of a constant negative curvature (by means of the cut and paste technique in the Lobachevsky space). In 3d this assumes that the simplest wormhole has throat sections in the form of a torus (see details in [7]). The subsequent cosmological evolution of such a space is governed by the standard Freedman equations $H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$, where $H = \frac{\dot{a}}{a}$ is the Hubble constant, ρ is the matter density, $k=-1$, and the metric takes the form $ds^2 = dt^2 - a^2(t)dl^2$, where dl^2 corresponds to the Lobachevsky space with a set of wormholes.

Thus, we see that our Universe can be rather far from the simple picture we use. Our phenomenological description based on the standard ΛCDM model works well enough, but meets some small inconsistencies which enforce us to add some exotic matter or use modifications of gravity. It is clear that the polarization of matter fields on the fractal topology is rich enough to be capable of explaining all exotic properties observed.

Consider the simplest model suggested by us [5] to demonstrate that the topological bias allows to mimic DM phenomena. When the gravitational field is rather weak, the Einstein equations for perturbations reduce to the standard Newton's law [8]

$$\frac{1}{a^2}\Delta\phi = 4\pi G\left(\delta\rho + \frac{3}{c^2}\delta p\right),$$

here a is the scale factor of the Universe, $\delta\rho$ and δp are the mass density and pressure perturbations respectively. The behavior of perturbations is determined by the Green function

$$\Delta G(x,x') = 4\pi\delta(r-r').$$

In the simple flat space the Green function defines the Newton's law $G_0 = -1/r$ (or for Fourier transforms $G_0 = -4\pi/k^2$). In the presence of wormholes due to polarization on throats the true Green function obeys formally to the same equation but with biased source (which is the topological bias or susceptibility)

$$\Delta G(x,x') = 4\pi\left(\delta(r-r') + b(r-r')\right).$$

In the case of a homogeneous gas of wormholes such a bias was evaluated first in [5] (see also more general consideration and details in [9]) and is given by

$$b(k) = 2n\overline{R}\frac{4\pi}{k^2}\left(\nu(k) - \nu(0)\right),$$

where n is the density of wormhole throats, $\overline{R}$ is the mean value of the throat radius, and $\nu(k)$ is the Fourier transform for the distribution over the distances between wormhole mouths. This function is normalized so that $\int \nu(X)d^3X = 1$. Thus the true Green function includes corrections and looks like

$$G(k) = \frac{-4\pi}{k^2(1-b(k))} \simeq \frac{-4\pi}{k^2}\left(1 + 2n\overline{R}\frac{4\pi\left(\nu(k)-\nu(0)\right)}{k^2}\right).$$

For the logarithmic correction (observed in galaxies) the distribution $\nu(k)$ should have the behavior $b(k) \sim k^\alpha$ with $\alpha \simeq 1$. This corresponds to the so-called fractal distributions of the type

$$\nu(k) = \exp\left(-A(ik)^\alpha - B(-ik)^\alpha\right).$$

In the case $\alpha < 2$ the dispersion is divergent, i.e., $\sigma = \frac{d^2}{dk^2}(\ln\nu(k))|_{k=0} \propto k^{\alpha-2} \to \infty$ and corrections to the Newton's law can be found from the decomposition of the characteristic function $\nu(k)$ as

$$\nu(k) - 1 = k^\alpha \sum C_n k^{\alpha n}.$$

Coefficients C_n reflect deviations from the standard Newton's law and may be interpreted as DM. They should be fixed from the observed distribution of DM which allow us to define the actual distribution of wormholes and restore the true Green function $G = \frac{-4\pi}{k^2}(C + Bk^{\alpha} + ...)$. As an empirical Green function (generalized susceptibility) we suggest the form

$$G_{emp} = \frac{-4\pi}{k^2\left(1 + (kr_0)^{-\alpha}\right)}.$$

We also point out that the logarithmic correction (observed in galaxies) corresponds to the value $\alpha \approx 1$ and $r_0 \sim 5Kpc$.

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Ionization-induced wavemixing of intense laser pulses

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The present work considers the ionization-induced wavemixing by studying the excitation of radiating low-frequency (terahertz) free-electron currents by two-color laser pulses in a gas. For the two-color pulses with frequency ratio of 2, these excitation phenomena and the resulting terahertz emission were previously studied theoretically and experimentally in the context of the well-known two-color scheme of laser-plasma terahertz generation [1-4]. Here we consider the arbitrary frequency ratio in the two-color pulse.

Using the *ab initio* quantum-mechanical simulations based on the solution of the 3D time-dependent Schrödinger equation, we find the dependence of the residual current density [4,5] to which the generated terahertz field is proportional on the frequency ratio. This dependence presents multiple resonant-like peaks with maxima at rational frequency ratios $a : b$ where a and b are natural and $a + b$ is odd. Such ratios commonly occur from synchronism conditions when the high-order wavemixing (rectification) in a centrosymmetric medium is considered. The dependences of peak strengths on the intensity of an one-color component of the pump pulse (when this intensity is not too big) are also similar to those typical for high-order wavemixing. To reveal the origin of these dependences, we use the semiclassical approach [4] and obtain analytical formulas which agree well with the quantum-mechanical calculations in the parameters regions corresponding to the tunnel ionization.

Our formulas indeed demonstrate that the excitation of low-frequency currents can be treated as ionization-induced multiwave mixing, and the number of mixed waves is determined by the steepness of the ionization probability as a function of ionizing electric field and therefore depends on the intensity of pump pulses. The latter distinguishes the phenomena considered from the common high-order wavemixing. Our quantum-mechanical and semiclassical calculations show that the terahertz generation efficiency can be of the same order for the common two-color pump laser pulses (with the frequency ratio $1 : 2$) and for that ones having uncommon ratios such as $2 : 3$. The idea of ionization-induced wavemixing can be useful not only for design and improvement of laser-plasma schemes of terahertz generation, but also for analyses of other ultrafast strong-field phenomena based on generation of harmonics and combination frequencies.

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Towards homoclinic bifurcations and complex dynamics in the system with a “figure-eight” of a saddle

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The problem of the effect of small time-periodic forces in an asymmetric Duffing–Van-der-Pol equation close to an integrable equation with a homoclinic “figure-eight” of a conservative saddle is considered.

Applying the Melnikov analytical method and numerical simulations, the existence regions of the rough homoclinic curve of the saddle periodic motion in the control parameters plane are established. The presence of such a curve specifies complicated behavior of solutions, in other words, it leads to chaos. The region in the parameters plane which has non-smooth boundaries is detected. Homoclinic bifurcations inside this region are studied.

The problem of the structure of boundaries which separate the attraction basins of stable fixed and periodic points of the Poincaré map is also discussed. We use an algorithm for calculating the fractal dimension of such boundaries. The presence of boundaries having fractal properties results in sensitive dependence of solutions on initial conditions close to these boundaries. It makes difficult predicting the final state of the system. It is established that the fractal dimension of attraction basins boundaries of stable regimes becomes more than topological one before the first homoclinic tangency of the invariant manifolds of the saddle periodic orbit.

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Hyperbolic chaos in model systems with ring geometry

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Ring systems of different nature with uniformly hyperbolic attractors are proposed. The operation of these systems is based on phase manipulation. Finite- and infinite-dimensional models are discussed.

The first ones are ring circuits of linear or self-sustained oscillators and nonlinear elements [1,2,3]. They are arranged in such a way that the phase of the excitations undergoes an expanding circle map transformation on each full revolution through the ring. This corresponds to the appearance of the Smale-Williams attractor in the phase space of the respective Poincaré map.

Examples of distributed systems describe active medias with Smale-Williams solenoids in Poincaré cross-section [4,5,6]. They are governed by partial differential equations with periodic boundary conditions which are the modifications of the Swift–Hohenberg equation and Brusselator model, as well as to the problem of parametric excitation of standing waves by the modulated pump. Proposed models demonstrate arise and decay of spatial wave patterns on a characteristic time period. Spatial Fourier harmonics of patterns interact with each other in such a way that phases of harmonics undergo an expanding circle map. This corresponds to the appearance of the Smale-Williams attractor in infinite-dimensional phase space. Reduced finite-dimensional models for distributed systems were obtained that describe only the most important modes. Their dynamics fits well with original systems.

Proposed systems were simulated numerically. Portraits of attractors in Poincaré cross-section were obtained. Attractors of these systems are clearly of Smale-Williams type. Iteration diagrams for phases topologically correspond to expanding circle map. Lyapunov exponents were evaluated by means of the Benettin algorithm. For each model the largest exponent is positive and close to Lyapunov exponent of the expanding circle map.

For attractors of the electronic circuit models and attractors of reduced models of distributed systems, a numerical test for hyperbolicity was conducted. Distributions of the angles between the stable and unstable subspaces on the attractors have been obtained. While zero angles are inherent to non-hyperbolic attractors, histograms demonstrate absence of them and confirm that attractors of proposed models are hyperbolic.

The considered systems may be implemented in electronics or optics. Attractiveness of systems with uniformly hyperbolic attractors in a frame of possible practical application of chaos is determined by their structural stability, or robustness: the generated chaos is insensitive to variations of parameters, imperfection of fabrication, technical fluctuations in the system, etc. Such systems can find applications in information technologies and cryptography since each trajectory on uniformly hyperbolic attractor corresponds to unique infinite sequence of symbols of finite alphabet.

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The dynamics of a system of phase oscillators depending on coupling strength and other parameters

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The current study aims to examine the dynamics of two coupled phase oscillators. We consider a general case of non-identical elements with their individual characteristics represented by parameters of the system. Thus, the main object is to describe regimes of the elements' activity depending on their characteristics and coupling strength. The current work is motivated by multiple problems of neuroscience where different types of neurons influence each other's behaviour by means of various interconnections [1]. Thus, it appears to be vital to examine how the coupling strength can influence the systems' dynamics. Not only can organisation of the connection between the elements change the systems' behaviour, but also in many cases it is considered to be more influential than individual parameters of the elements[2]. Similar studies are presented in these papers [3-7].

In this study we considered the following system of equations where ϕ_1, ϕ_2 are phases of two oscillatory elements, γ_1, γ_2 - individual characteristics of the elements, d - coupling strength. Both γ_1 and γ_2 vary from *-2* to *2*.

$$\begin{cases} \dot{\phi_1} = \gamma_1 - \sin\phi_1 + d\sin(\phi_2 - \phi_1) \\ \dot{\phi_2} = \gamma_2 - \sin\phi_2 + d\sin(\phi_1 - \phi_2) \end{cases} \tag{19}$$

In order to facilitate better understanding of the system's dynamics we obtained a series of bifurcation diagrams in the plane (γ_1,γ_2) for different values of the coupling strength. These are presented in Fig. 12.As one can see here, there are several regimes of activity in system (19). Firstly, there can possibly exist, depending on the value of the coupling strength d, three types of 'steady-state' domains - E_2, E_4, E_6 - with 2, 4 and 6 steady stated respectively. There are 6 equilibria in domain E_6: a stable node, 3 saddles and 2 unstable nodes. There are 4 equilibria in domain E_4: a stable node, 2 saddles and an unstable node. There are 2 equilibria in domain E_2: a stable node, a saddle and an unstable node. Phase portraits are shown in Fig. 13. The greater is d, the bigger all these domains become. At the moment of passing from E_2 to E_4

the saddle-node bifurcation occurs. Similarly, it occurs when we pass from E_4 to E_6. Secondly, there are three other regimes with no steady states in the system. In domain D_s a periodic rotation mode for both variables is observed. This domain stands for synchronisation. In D_q a quasi-periodic regime is observed. Both phases increase without bound in this domain. In D_p we have periodic rotations for one of the variables (for ϕ_1 for the domains where $\gamma_1 > \gamma_2$ and for ϕ_2 elsewhere). One of the phases is unbounded here. Another one oscillates within specific range. Clearly, strong coupling results in larger areas of synchronisation.

To sum up, coupling strength plays a crucial role in determining a possible number of steady states in the system of two coupled phase oscillators. Individual characteristics of the oscillators were proved to decide its actual behaviour (including the number of steady states). Depending on exact values ... cs coupling strength can either facilitate an un-limited increase ... d ... notion, coupling strength has a s ... r ... r.

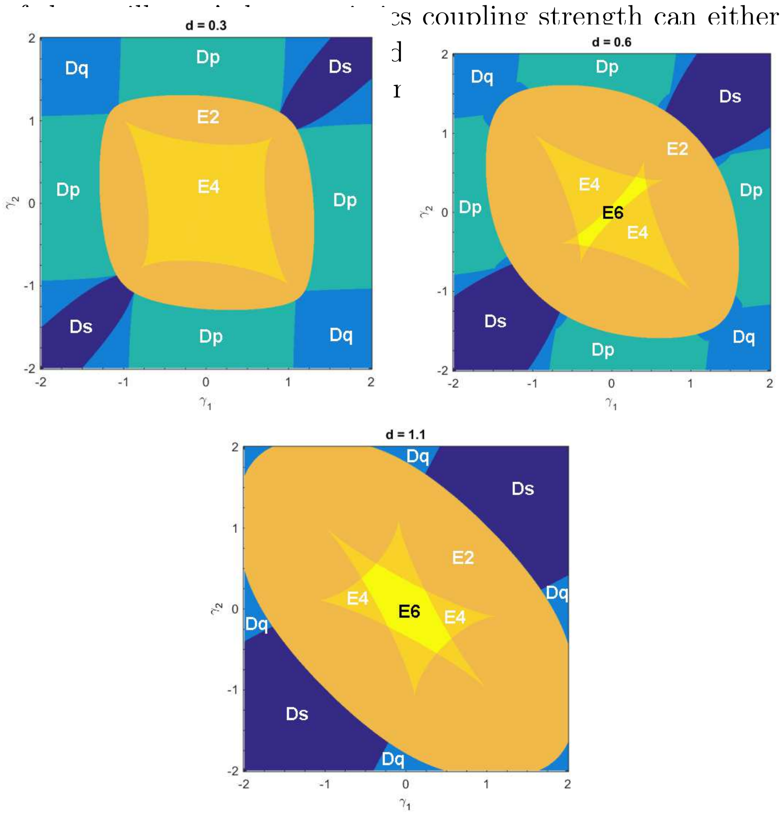

Figure 12: Bifurcation diagrams for $d = 0.3$, $d = 0.6$ and $d = 1.1$. There are 6 equilibria in domain E_6: a stable node, 3 saddles, 2 unstable nodes. There are 4 equilibria in domain E_4: a stable node, 2 saddles, 1 unstable node. There are 2 equilibria in domain E_2: a stable node, a saddle and an unstable node. In D_s a periodic rotation mode for both variables is observed. In D_q a quasi-periodic regime is observed. In D_p we have periodic rotations for one of the variables (for ϕ_1 for the domains where $\gamma_1 > \gamma_2$ and for ϕ_2 elsewhere).

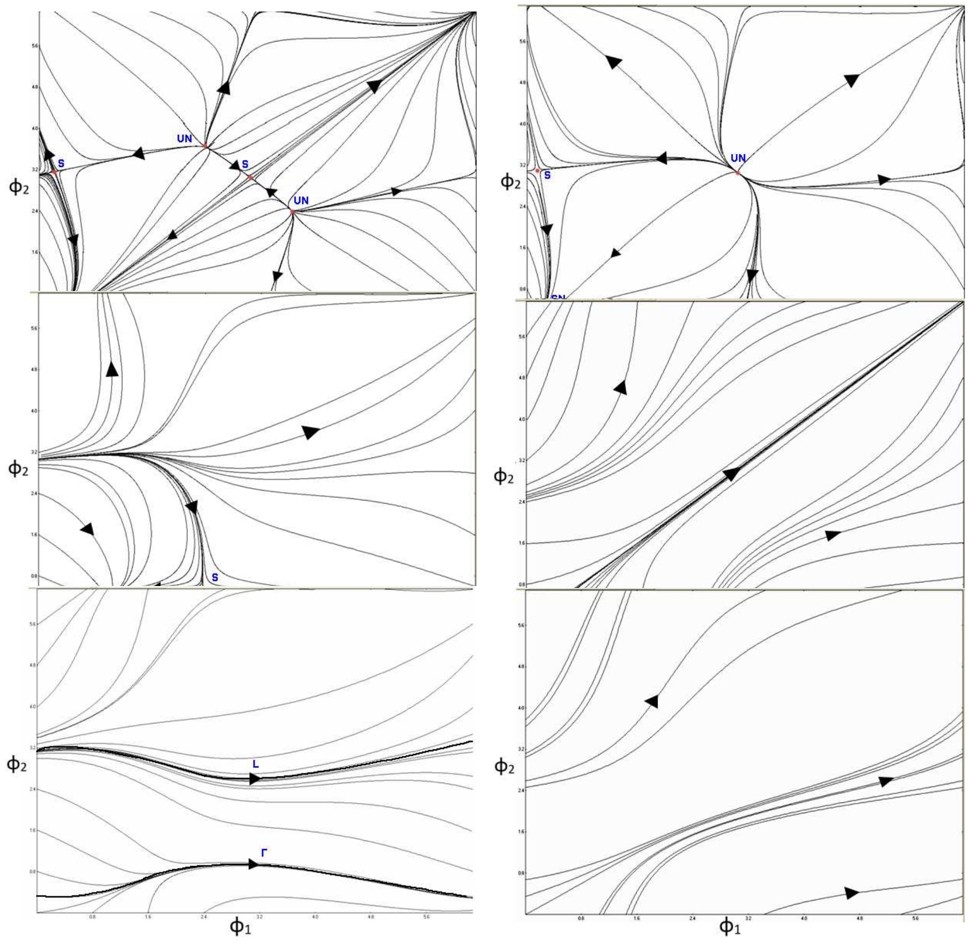

Figure 13: Phase portraits of system (19) for $d = 0.6$. Individual characteristics $\gamma_{1,2}$ are chosen so that all types of activity observed in Fig. 12 are represented here. Each portrait corresponds to a certain domain in Fig. 12a: **??** - E_6; **??** - E_4; **??** - E_2; **??** - D_s; **??** - D_p; **??** - D_q. Both γ_1 and γ_2 are positive here.

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Nonlinear dynamics of the number of different age specialists in the regional economy

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The questions about the stability of socio-economic systems are very topical in modern economics. One such system is the regional labor market. A variety of socio-economic interactions between employed population can provoke a strong polarization of the role of age groups on the labor market, which is likely to have a negative impact on quality of life, economic development and attractiveness of the region.

In modern science, the interaction of various objects is studied at a high level with the use of nonlinear mathematical models (Zhang, 1996; Weidlich, 2000; Milovanov, 2001). Using shaped approaches to the study of complex interaction of objects, we propose to examine the interaction of different age professionals specialists in the regional economy in the key of nonlinear dynamics.

It was found that the complex three-dimensional structure in the phase space is formed through a cascade of period doubling cycle or by a gradual noise on the limit cycle, which demonstrates the system's sensitivity to small deviations of the initial conditions and the fundamental impossibility of medium-and long-term forecasting.

Unpredictable changes in the number of different age specialists of can be explained by unfavorable situation on the labor market: the migration flow prevails over the outflow of only one cohort, all age groups are characterized by a significant influx of economically inactive. It should be noted that complex periodic modes appear in a fairly narrow areas of the parameter space and have varying degrees of stability to change specific parameters. In the applied aspect, it means that to eliminate or smoothing fluctuations in the labor market can be regulated by the bifurcation parameter of the system.

The model equations used to describe the dynamics of the interaction of the employed population in the Jewish Autonomous Region (JAR) of the three age groups: 16-29, 30-49 and 50 and over, according to 1997-2010.

The proposed model reveals the nonlinear effects in the development of the labor market. It's necessary to considered in the management of the region and the development of forecasts.

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Clustering in the coupled Ricker population model

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In various sciences there is a problem of studying the features coupled oscillating elements dynamics. For example these are systems of coupled neurons, lasers, alternating current generators or migration-related populations. One kind of this model is a system of coupled logistic maps or coupled map lattice (CML). The single map of CML demonstrates the transition to chaos through period doubling.

To describe the dynamics of biological population or metapopulation the each patch is modeled a logistic map (Ricker model or Ricker map) and relationship between neighboring populations (migration flows) is a dissipative coupling between neighboring oscillators on plane lattice. As result the coupling between them apparently can not be global, leading to the complex dynamics mode. For instance besides the phenomena of multistability and synchronization there is a “boundary effect” which makes it impossible to complete synchronization, especially for large values of the migration coefficient (coupling force). As consequence the initially identical and symmetrically coupled maps in different clusters may fluctuate with the different amplitudes, phases or periods.

In this abstract the phenomenon of clustering and multistability in coupled Ricker population model are studied.

It is shown the forming of clusters is complex depends on an initial distribution of individuals in flat area. It is described the bifurcation scenario of two non equal clusters formation correlated clusters with ordered phase. As result it is shown its attraction basins of these clutters coincide partially with basins of antiphase modes in system of two non-symmetrical and not identical coupled Riker maps. It is found the phase space consists of a large number of embedded in each other basins of similar clustering phase. These modes are kind of transition states between coherent mode and phase with two equal clusters. In addition to these modes’s domains of attraction the basins of three-cluster states are found. Basins of these modes are union of attraction domains of two-cluster phases with a different number of occupation numbers.

Moreover the dynamic of two coupled non-identical Ricker maps on the average approximates the dynamic of two non equal clusters. It is shown the differences between each of these maps depend on clusters size and number of direct connections between clusters. The study of non-identical maps Ricker dynamic modes makes it possible to suggest what dynamic modes are possible for two unequal clusters. As a result it is found the ordered phase with a fixed size and location of clusters has a multistable nature. It means it can be implemented in several ways. These ways are not only the order of the system states, i.e. the phase of oscillations, but a fundamentally different type of dynamics: various cycle lengths and location attractor in the phase space. Furthermore the formation of two or more clusters has the same mechanism as bifurcation of the generation asynchronous modes in the approximating system. But further bifurcations do not always coincide.

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Parametric generators of chaos

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Several schemes of parametric generators of chaotic oscillations are suggested and analyzed. Among them there are systems with Lorenz-type attractor and attractors represented by a kind of Smale-Williams solenoid in Poincaré map [1-3]. Beside the mathematical models and numerical computations, concrete electronic schemes are considered, and results of their simulation using software product Multisim are presented and discussed. The work is supported in part by RFBR grants No 14-02-00085 and 15-02-02893 and by the grant for leading scientific schools NSh-1726.2014.2.

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Dynamics and bifurcations in a simple quasispecies model of tumorigenesis

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Cancer is a complex disease and thus is complicated to model. However, simple models that describe the main processes involved in tumoral dynamics, e.g., competition and mutation, can give us clues about cancer behaviour, at least qualitatively, also allowing us to make predictions. Here, we analyse a simplified quasispecies mathematical model given by differential equations describing the time behaviour of tumor cell populations with different levels of genomic instability. We find the equilibrium points, also characterizing their stability and bifurcations focusing on replication and mutation rates. We identify a transcritical bifurcation at increasing mutation rates of the tumor cells. Such a bifurcation involves a scenario with dominance of healthy cells and impairment of tumour populations. Finally, we characterize the transient times for this scenario, showing that a slight increase beyond the critical mutation rate may be enough to have a fast response towards the desired state (i.e., low tumour populations) by applying directed mutagenic therapies.

This is a joint work with Josep Sardanyés[†] and Vanessa Castillo[‡].

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On topological classification of diffeomorphisms of 3-manifold with one-dimensional surface basic sets

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In this report we consider a class of diffeomorphisms of 3-manifold, such that each diffeomorphism fron this class is a locally direct product of a DA-diffeomorphism of 2-torus and rough diffeomorphism of the circle. We find algebraic criteria for topological conjugacy of the systems. It is proved that the class of topological conjugacy of such diffeomorphism is completely determined by combinatorial invariants, namely hyperbolic automorphism of the torus, a subset of its periodic orbits, the number of periodic orbits and the serial number of the diffeomorphism of the circle.

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Multistable regimes in the motif of Rulkov maps with inhibitory couplings

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Using three coupled Rulkov maps, we construct here a minimal motif of three neurons mutually coupled by inhibitory synapses. Rulkov discrete neuron model was introduced in [1] and later studied and developed in [2]-[6]. Our modeling takes into consideration basic principles of chemical coupling: (i) presence and absence of postsynaptic element activity depend on the presynaptic element activity level, and (ii) all interactions between cells are inertial because of chemical processes in neurotransmitters. This model is discrete and, therefore, it is rather easy to numerical analysis.

We study different types of activity that can be generated in this motif by governing coupling parameters.

In order to built the described motif we take the neuron-like Rulkov map as an isolated element:

$$\begin{aligned} x_{n+1} &= f(x_n, z_n, y_n) \\ y_{n+1} &= y_n + \mu(-x_n - 1 + \sigma) \\ z_{n+1} &= x_n \end{aligned} \tag{20}$$

where x is a fast variable that qualitatively describes fast ionic currents (Na^+ and K^+) in the cell, and, more generally, corresponds to membrane potential. In the one-dimensional map $x_{n+1} = f(x_n, y_n, z_n)$ f is a discontinuous function

$$f(x,y,z) \; = \; \begin{cases} \alpha/(1-x)+y, & x \leq 0, \\ \alpha+y, & 0 < x < \alpha+y \; and \; z \leq 0, \\ -1, & x \geq \alpha+y \; or \; z > 0. \end{cases} \tag{21}$$

It is constructed in a way to reproduce different regimes of neuron-like activity, such as spiking, bursting and silent regimes.

Variable y corresponds to slow ionic currents such as Ca^{2+}. Its equation sets a nonlinear feedback coupling and makes some nonlinear transient processes possible to be modeled.

Variable z corresponds to the dependency between x_{n+1} and x_{n-1}.

According to the neuron-like dynamics of the map it is possible to construct a low-dimensional model of a neuron that are iterating with a time step congruent to the spike duration. The single element model is able to demonstrate different types of neuron-like activity depending on the parameters α which is a control parameter of the map, and σ which sets the non-perturbed state of the three-dimensional map.

In our modeling we take values of $\alpha = 3.9$ and $\sigma = 1$ from the parameter region so that the element demonstrates a regular tonic spiking regime. $\mu = 0.001$ is a small constant that provides slow changes of the variable y. In the phase space for an individual element there is a periodic point of period 4 for these parameters.

In order to build a plausible model for the inhibitory coupling principle, we use as coupling term an additional term in the right parts of equations (20)

$$I_n^{ji} = \gamma_{ji} I_{n-1}^{ji} + g_{ji}(x_{rp} - x_n^i)\xi(x_n^j); \tag{22}$$

multiplied by different constants for x_{n+1} and y_{n+1}. Here j is the presynaptic element, and i is the postsynaptic one. The parameter γ_{ji} is a relaxation time of the synapse, $0 \leq \gamma_{ji} \leq 1$. It defines the part of synaptic current which preserve as in the next iteration. g_{ji} corresponds to the strength of synaptic coupling and are the governing parameters of the system, $g_{ji} \geq 0$. x_{rp} is a reversal potential that determines the type of the synapse ($x_{rp} = -1.5$ corresponds to the inhibitory synapse, and $x_{rp} = +1$ - to the excitatory one). $\xi(x)$ is a step function

$$\xi(x) \; = \; \begin{cases} 1, & if \; x > x_{th}, \\ 0, & else, \end{cases} \tag{23}$$

with the threshold value x_{th} that implements the principle of inhibitory coupling. If the value of the membrane potential x exceeds the threshold value x_{th}, then the presynaptic elements suppress all activity except subliminal oscillations in the postsynaptic element in case of nonzero coupling.

Finally, the discrete equations that describe a motif consisting of 3 Rulkov elements with synaptic reciprocal couplings are

$$\begin{array}{l} x_{n+1}^i = f(x_n^i, x_{n-1}^i, y_n^i + \frac{\beta_{syn}}{2}\sum_{j\neq i}(I_n^{ji}), \alpha_1); \\ y_{n+1}^i = y_n^i + \mu_i(-x_n^i - 1 + \sigma_i + \frac{1}{2}\sum_{j\neq i}(I_n^{ji})); \\ z_{n+1}^i = x_n^i; \\ i = 1,2,3. \end{array} \tag{24}$$

We assume $g_{12} = g_{23} = g_{31} = g_1$, $\gamma_{12} = \gamma_{23} = \gamma_{31} = \gamma_1$, $g_{21} = g_{32} = g_{13} = g_2$, $\gamma_{21} = \gamma_{32} = \gamma_{13} = \gamma_2$ in our study. Also we assume $x_{rp} = -1.5$, $\beta_{syn} = 0.0001$.

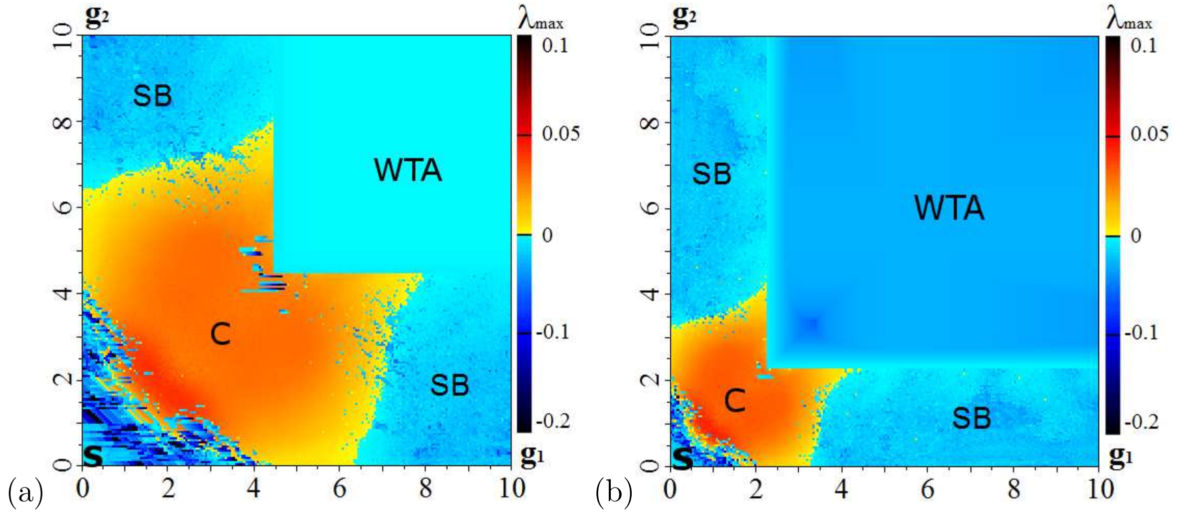

Figure 14: LLE charts of the system (24). Different regumes of neuron-like activity are marked by the following abbreviations: SB - the region of sequential bursting dynamics, S - the region of spiking activity, WTA - the region of winner-takes-all regimes, C - chaotic region. (a) $\gamma_1 = \gamma_2 = 0$ (b) $\gamma_1 = \gamma_2 = 0.5$

In order to study different neuron-like regimes in this ensemble we built charts of the largest Lyapunov exponent (LLE) and on the parameter plane $P = (g_1, g_2) : g_i \in [0, 10]$, divided into 200×200 nodes.

On the constructed LLE charts (fig. 14) we mark different regimes (depending on the LLE, level of activity of each element, interspike intervals and presence of burst) using the following abbreviations:
(i) SB – sequential bursting activity;
(ii) S – regular spiking activity;
(iii) C – chaotic spiking activity;
(iv) WTA – winner-take-all activity.

We studied bifurcation transition from multistable regimes of winner-take-all activity to chaotic activity and sequential switching activity using analysis of calculated multipliers of periodic points (see Fig.15). We performed the following experiment: fix the coupling value $g_2 = 7$, and start to decrease g_1 from the initial value $g_1 = 7$. We found that in this case triplets of periodic points of periods 4, 5 and 6 sequentially bifurcates via a subcritical Neimark-Sacker bifurcation, when the coupling parameter reaches the threshold values for each triplet of periodic points, and then hard transition to chaos occurs.

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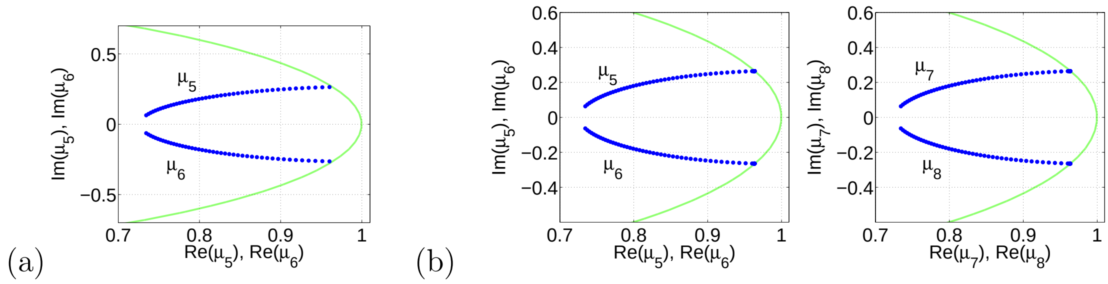

Figure 15: (a) Subcritical Neimark-Sacker bifurcation of a triplet of periodic points: two complex-conjugated multipliers μ_5 and μ_6 (blue dots) of the periodic point of period 4 cross unit circle (green curve), $g_2 = 7$, g_1 variates from $g_1 = 7$ to $g_1 = 4.99$ (b) Double subcritical Neimark-Sacker bifurcation of the triplet of periodic points of period 4: two pairs of complex conjugated multipliers (μ_5, μ_6) and (μ_7, μ_8) (blue dots) of the corresponding periodic point cross unit circle (green curve), coupling values variate from $g_1 = g_2 = 7$ to $g_1 = g_2 = 4.99$. In both cases other multipliers remain inside unit circle.

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Finiteness and existence of attractors and repellers on sectional hyperbolic sets

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We study small perturbations of a sectional hyperbolic sets of vector fields on compact higher dimensional manifolds. Particularly, the aim is to research two very important related problems, namely, how many attractors and repellers can arise from small perturbations and, also, the possible appearance of repellers from small perturbations. The motivations come from the previous result in dimension three [1], [7] in which an upper bound in terms of the number of singularities is obtained but for transitive or nonwandering flows. Another motivation is the recent paper by the author [6] where the same results for transitive flows were obtained in higher dimensions. A further motivation is the well known Anomalous Anosov flow [5] beside [3], which are examples of connected sectional hyperbolic sets (with or without singularities) containing nontrivial repellers. Here we will remove both transitivity and nonwandering hypotheses in order to obtain the finitude in a robust way of attractors and repellers for higher dimensional sectional hyperbolic sets. Moreover, we prove the non-existence of repellers for any perturbation of a connected sectional hyperbolic sets (with singularities) contained in the nonwandering set. As consequence, Main Theorem also provides a corollary related to one of the Bonatti's conjectures [2]. Being more precise, Palis conjectured in [8] that generic diffeomorphisms far from homoclinic tangencies have only finitely many sinks and sources. Such a conjecture is true in the surface case by Pujals-Sambarino [9]. More recently, Bonatti stated a slightly stronger conjecture, namely, generic diffeomorphisms that are far from homoclinic tangencies have only finitely many attractors and repellers [2]. In our view, it is natural to consider even the flow version of Bonatti's conjecture, namely, if a generic flow far from homoclinic tangencies has only a finite number of attractors and repellers.

In this direction, [4] proved in the C1 topology that every three dimensional flow can be accumulated by robustly singular hyperbolic flows, or by flows with homoclinic tangencies. This fundamental tool together with ours implies positive answer in dimension three.

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Actions of symplecting groups of diffeomorphisms of orientable surfaces on spaces of smooth functions

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Let M be a closed orientable surface and $D(M)$ be the group of diffeomorphisms of M acting from the right on $C^\infty(M)$ by the rule: the result of the action of $h \in D(M)$ on $f \in C^\infty(M)$ is the composition $f \circ h : M \to R^1$. Let $S(f) = \{f \circ h = f \mid h \in D(M)\}$ and $O(f) = \{f \circ h \mid h \in D(M)\}$ be respectively the stabilizer and the orbirt of $f \in C^\infty(M)$. Denote by $S_{\mathrm{id}}(f)$ the path component of id_M in $S(f)$ and by $O_f(f)$ the path component of f in $O(f)$. In a recent series of papers the author described the homotopy types of $S_{\mathrm{id}}(f)$ and $O_f(f)$ for a large class of smooth functions on M which includes all Morse functions.

Let ω be any symplectic 2-form on M. Then one can consider the restriction of the above action of $D(M)$ on the groups $Symp(M, \omega)$ of symplectic diffeomorphisms of M. For $f \in C^\infty(M)$ let $S(f, \omega)$ and $O(f, \omega)$ be the corresponding stabilizer and orbit of f with respect to $Symp(M, \omega)$. Then $S(f, \omega) = S(f) \cap D(M)$ and $O(f, \omega) \subset O(f)$. Similarly, let $S_{\mathrm{id}}(f, \omega)$ the path component of id_M in $S(f, \omega)$ and by $O_f(f, \omega)$ the path component of f in $O(f, \omega)$.

Theorem *Let $f : M \to R^1$ be a C^∞ Morse function. Then there exists a symplectic 2-form ω on M such that both inclusions $S_{\mathrm{id}}(f, \omega) \subset S_{\mathrm{id}}(f)$ and $O_f(f, \omega) \subset O_f(f)$ are weak homotopy equivalences.*

Renormalization and Universality for Rotation Sets in Lorenz-like Systems

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Geometric models of Lorenz-like systems as well as models in the form of countable topological markov cchains are considered. By using rotation sets of the models and of their renormalizations we study behavior of rotation sets in one-parameter families of multidimensional perturbations of one-dimensional maps of Lorenz type. More precisely, let

$$\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0, \; n \in \mathbf{Z},$$

be a difference equation of order m with parameter λ. It is asuumed that the non-perturbed operator Φ_{λ_0} depends on two variables, i.e., $\Phi_{\lambda_0}(y_0, \ldots, y_m) = \psi(y_N, y_M)$, where $0 \leq N, M \leq m$ and ψ is a piecewise monotone piecewise C^2-function. It is also assumed that for the equation $\psi(x, y) = 0$, there is a branch $y = \varphi(x)$ which represents a one-dimensional Lorenz-type map. We prove approximation results for the problem on continuous dependence of the rotation set under multidimensional perturbations. Numerical results show universality phenomena in the bifurcation structure responsible for birth of nontrivial rotation intervals with respect to the maps and also to their renormalizations. Our technique is based on approximations of topological entropy and maximal measures represented by countable topological Markov chains [1] and also, on continuation of chaotic orbits for perturbations of singular difference equations [2].

Besides, we compare the results for behavior of nonwandering orbits of the systems under consideration with ones of generalized polynomial Hénon maps.

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On classification of Morse-Smale systems with few non-wandering points

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We consider Morse-Smale systems whose non-wandering set consists of three fixed points. We formulate necessary and sufficient conditions for such systems to be conjugacy.

This work was supported by the Russian Foundation for Basic Research, Grants 15-01-03687 and 13-01-12452-ofi-m.

On transitory systems

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We consider the two-dimensional non-autonomous systems one whose time-dependence is confined to a bounded interval. Following [1], such systems we call transitory.

We consider a subclass of transitory systems that are close to nonlinear conservative. As an example the system

$$\begin{aligned} \dot{x} &= y - \omega f(t), \\ \dot{y} &= x - x^3 + \varepsilon(p - x^2)y \end{aligned} \tag{25}$$

is investigated. Here ε is a small parameter, p, ω are parametres. Transition function $f(t)$ is the smooth function, satisfying the condition

$$f(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t \geq \tau. \end{cases}$$

The system (1) at $t \leq 0$ takes place in the flutter problem [2,3]. In the conservative approximation of this problem ($\varepsilon = 0$) we identified a quantitative evaluation (a measure) of transport between coherent structures. In the nonconservative approximation we consider the impact of transitory shift to setting of one or another attractor. We give probabilities of changing a mode (stationary to auto-oscillation). Also we point out to the possibility of oscillation stabilization as a result of the transitory shift.

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Phase diffusion in unequally noisy coupled oscillators

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We consider the dynamics of two directionally coupled unequally noisy oscillators, the first oscillator being nosier than the second oscillator. Examples include spontaneous stochastic oscillations of sensory hair cells and synaptically coupled neurons. We derive analytically the phase diffusion coefficient of both oscillators in a heterogeneous setup (different frequencies, coupling coefficients and intrinsic noise intensities) and show that the phase coherence of the second oscillator depends in a non-monotonic fashion on the noise intensity of the first oscillator: as the first oscillator becomes less coherent, i.e. worse, the second one becomes more coherent, i.e. better. This surprising effect is related to the statistics of the first oscillator which provides for the second oscillator a source of noise, which is non-Gaussian, bounded, and possesses a finite bandwidth.

Features of dynamics of the implicit maps. Conservative and near conservative case

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In the theory of chaos it is well-known the special class of dynamical systems - the discrete maps. There are reversible in time maps, when each point in phase space has only one preimage, and nonreversible naps, when several preimages may be occur. One can imagine such a discrete model in which there is several solutions in forward as in backward time. These systems have an implicit evolution operator

$$g(z_{n+1}, z_n) = 0. \tag{26}$$

The maps of such type are abstract models, as well as the explicit but nonreversible maps. In work [S.R. Bullett et al //Physica 19D, 1986, P.290] an implicit map of the complex plane with special unitary symmetry is described. The conservative trajectories representing a stochastic web are found in the phase p lane of this system. In present work, we make an attempt to study the transition from an nonreversible Mandelbrot map to an implicit one, until conservative limit.

The work was supported by RFBR (projects 14-02-31067)

Attractors of skew products with circle fiber

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Partially hyperbolic skew products over Anosov diffeomorphisms and their "younger brothers", step skew products over topological Markov chains, provide interesting dynamics even for one-dimensional fiber. For the interval fiber there are strange examples in border-preserving case : intermingled basins [1] and thick (i.e. having positive but not full measure) attractor [2]. In non-border-preserving case there is a finite collection of alternating attractors and repellers, each of them is a bony graph [3]. The Milnor attractor is the smallest closed subset, containing ω-limit sets of almost all points. It is unknown whether the Milnor attractor is the union of these bony graphs (or, equivalently, whether it is asymptotically stable).

We will discuss the following result in the circle fiber case: the Milnor attractor is Lyapunov stable and not thick; either the skew product is transitive or the nonwandering set has zero measure. Main ingredients of the proof are the semicontinuity lemma and the fact that ω-limit set of a generic point is saturated by unstable leaves.

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Criteria of the birth of Lorenz attractors

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Generic criteria of the birth of Lorenz attractors for systems with two homoclinic loops were formulated in [1]. Based on them, explicit conditions on the parameters of the homoclinic orbits were developed by L.P. Shilnikov in [2, 3]. In the present work these conditions were extended to the situation when, unlike the Lorenz-like systems, there is no symmetry present. In particular, the considered cases include the so-called *semi-orientable Lorenz attractors.*

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On topological classification of structurally stable dynamical systems

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The results were obtained in collaboration with V.Grines and S. Van Strien.

It is a step towards the complete topological classification of Ω-stable diffeomorphisms on an orientable closed surface, aiming to give necessary and sufficient conditions for two such diffeomorphisms to be topologically conjugate without assuming that the diffeomorphisms are necessarily close to each other. In this paper we will establish such a classification within a certain class Ψ of Ω-stable diffeomorphisms defined below. To determine whether two diffeomorphisms from this class Ψ are topologically conjugate, we give (i) an algebraic description of the dynamics on their non-trivial basic sets, (ii) a geometric description of how invariant manifolds intersect, and (iii) define numerical invariants, called moduli, associated to orbits of tangency of stable and unstable manifolds of saddle periodic orbits. This description determines the *scheme* of a diffeomorphism, and we will show that two diffeomorphisms from Ψ are topologically conjugate if and only if their schemes agree.

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Two-parameter bifurcation study of the regularized long-wave equation

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We perform a two-parameter bifurcation study of the driven-damped regularized long-wave equation by varying the amplitude and phase of the driver. Increasing the amplitude of the driver brings the system to the regime of spatiotemporal chaos (STC), a chaotic state with a large number of degrees of freedom. We identify four distinct routes to STC; they depend on the phase of the driver and involve boundary and interior crises, intermittency, the Ruelle–Takens scenario, the Feigenbaum cascade, an embedded saddle-node, homoclinic and other bifurcations.

Feynmanons in gravitational waves in fluid

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In this report we consider linear gravitational waves in incompressible infinitely deep fluid on the plane z = 0 corresponding to unperturbed surface of fluid. In this case one proves to find equation of hyperbolic type for potential of velocity of fluid at z = 0. By the same way as in report [1] we reduce this equation to the vector Schrödinger type equation with two spatial variables. Green's matrix of this vector equation is expressed by Feynman integral too. This path integral with four-dimensional phase space has fractal structure of its trajectories described in detail in paper [2]. Nonlocal quantum field theory for two-dimensional potential of velocity of fluid also is presented. In accordance with our general line [1, 2] we introduce the new kind of feynmanon as object moving along fractal trajectories in considered Feynman integral. We have called this quasiparticle by "surface hydron". The results of this work can be easily extended on a number of more complicated situations for linear gravitational waves namely on the case of incompressible infinitely deep fluid with surface tension, on the case of incompressible fluid with finite deep, on the case of layers of incompressible fluids with different densities and so forth.

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Plane waves, Lie groups and Feynman integrals

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Using the representation of plane wave in n-dimensional space by means of Feynman integral [1] we show that symmetry of this wave is closely connected with semidirect product of real symplectic group Sp(2n,R) and Heisenberg-Weyl group H(n).

If $n = 4$ then phase space of corresponding Feynman integral is eight-dimensional space. It means that in this case four momenta and four coordinates one can consider as eight components of octonion. It is well-known that five exceptional simple Lie groups $G2, F4, E6, E7$ and $E8$ are related to octonions [2]. Furthermore group $G2$ is group of symmetry for octonionic algebra. On the other side simple Lie groups $G2$ is necessary in theory of strings in order to reduce eleven dimensions of M-theory to four dimensions of our space-time [2].

If one introduce eight-dimensional lattice in the phase space then random walk on this lattice is subset of set of phase trajectories in the Feynman integral. In particular an example of such trajectory is eight-dimensional Peano curve filling eight-dimensional hypercube [3].

As in three-dimensional case [4] dynamics of momenta may obey to succession map. If this map satisfies to conditions of Williams-Hatchinson theorem [3] then momenta form four-dimensional fractal set. In succession map may take place chaotic behaviour too. On the other hand four momenta one can consider as four components of quaternion. If we split four components of this quaternion on two complex variables then our succession map for momenta can be reduced to holomorphic map possessing by very untrivial dynamics.

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Multistability in quasiperiodically driven Ikeda map

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It is well known that dynamical systems with weak dissipation can demonstrate a great number of attractors [1]. In this work mechanisms of phase space structure changes for the system with weak disipation while the quasiperiodical influence to the system is induced have been investigated using the Ikeda map [2].

The investigated map is given by

$$E_{n+1} = A(1 + \varepsilon \sin(\Omega \cdot \varphi \cdot n)) + BE_n \exp(i|E_n|^2 + i\varphi),$$

where A is control parameter, B is parameter of dissipation, ε is amplitude of external influence, Ω is frequency of influence, φ is phase.

In the work the structure of coexisting attractors and their evolution while ε is changed in the case of weak dissipation are investigated. It is shown that the number of coexisting attractors decreases in comparison with the case of absence of external influence. It occurs due to the finite size of torus attractor in contrast to periodical attractor. The attractor number dependence on the ε is studied for different values of A. The evolution of attractor basins boundaries while quasiperiodical influence is appended has been studied.

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Bifurcations of first integrals in the Kowalevski – Sokolov case

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The phase topology of the integrable Hamiltonian system on $e(3)$ found by V.V.Sokolov (2001) [1] and generalizing the Kowalevski case (1889) [2] is investigated. The generalization contains, along with a homogeneous potential force field, gyroscopic forces depending on the configurational variables. In this talk we consider the problem with the following Hamiltonian

$$H = \frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \varepsilon_1(\alpha_3 M_2 - \alpha_2 M_3) - \varepsilon_0 \alpha_1.$$

The first integral additional to H found in [1] proves the complete Liouville integrability of the family of Hamiltonian systems on $\mathcal{P} = \{\mathbf{M} \cdot \boldsymbol{\alpha} = 2\ell, \boldsymbol{\alpha}^2 = a^2\}$. This integral can be written in the form

$$
\begin{aligned}
K &= \left[\frac{1}{4}(M_1^2 - M_2^2) + \varepsilon_1(\alpha_2 M_3 - \alpha_3 M_2) - \varepsilon_1^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \varepsilon_0\alpha_1\right]^2 + \\
&+ \left[\frac{1}{2}M_1 M_2 + \varepsilon_1(\alpha_3 M_1 - \alpha_1 M_3) + \varepsilon_0\alpha_2\right]^2 .
\end{aligned}
$$

In mechanics, much attention is paid to the investigation of specific motions of mechanical systems (including integrable cases), to their analytical description and stability features. In the talk we obtain a complete analytical description of all relative equilibria of the Kowalevski – Sokolov top. We calculate their types and establish their stability character. We show all Smale diagrams existing in this problem, calculate the Morse indices of the reduced energy and describe the topology of all iso-energy manifolds. The set of critical points of the complete momentum map is represented as a union of critical subsystems; each critical subsystem is a one-parameter family of almost Hamiltonian systems with one degree of freedom. For all critical points we explicitly calculate the characteristic values defining their type.

The information obtained gives us a possibility to classify all kinds of iso-integral bifurcation diagrams equipped with the notation of bifurcations at the edges and the number of regular Liouville tori attached to each chamber. This completely defines the so-called rough topology of the system. In turn, the knowledge on the rough topology provides the final description of relative equilibria by pointing out the topological structure of their saturated four-dimensional neighborhoods in the phase spaces of the reduced systems in those cases when this structure is not uniquely defined by the analytically calculated type of a critical point. Finally, having constructed the Smale – Fomenko diagrams (the separating set for the rough topological invariants of reduced systems), we present all 25 Fomenko graphs of the Kowalevski – Sokolov top.

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Strange nonchaotic attractors in quasiperiodically driven Ikeda map with weak dissipation

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It is known that the quasiperiodically driven systems demonstrate some specific features of dynamics, i.e. the realisation of strange nonchaotic attractors (SNA). The dynamics of quasiperiodically driven dissipative systems is well studied now [1]. It seems interesting to consider the peculiarities of these phenomena in the systems with weak dissipation, in particular to reveal the SNAs and corresponding regions in the parameter plane. Recently [2] it was shown that SNA exists in the map with weak dissipation but no results consider the size of corresponding region on the parameter plane and its dependence on dissipation parameter is known.

In this work we consider the well-known Ikeda map [3] driven by signal with incommensurate frequency:

$$z_{n+1} = A(1 + \varepsilon \sin 2\pi\theta_n) + Bz_n e^{i|z_n|^2}, \theta_{n+1} = \theta_n + w.$$

The frequency ratio w was taken equal to golden ratio $w = (\sqrt(5) - 1)/2)$ and the structure of the "nonlinearity parameter A - the driving amplitude ε" plane was studied for different dissipation values B by calculation the Lyapunov exponents.

The identification of SNAs was based on the values of largest Lyapunov exponent and visual analysis of its structure both with the rational approximants [1] method. It was revealed that the band of parameter A values in which the SNA exists decreases significantly with the decrease of dissipation.

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Experimental study of transition to the robust chaos in the Kuznetsov generator

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An example of a dynamical system of ODEs demonstrating the regime of robust hyperbolic chaos was first introduced in the work of Kuznetsov [1], and later the radiotechnical device demonstrating such type of dynamics was constructed [2]. Such generator of chaos could be very useful for applications due to the robustness of its dynamics.

In the present work we introduce another example of the radiotechnical Kuznetsov generator and investigate its parameter plane experimentally at frequencies of 50 kHz.

In presented system Smale-Williams-like solenoidal attractor exists in the wide domain on the parameter plane. We calculate different characteristics for the trajectories in this domain, particularly, the iteration diagram for the phase of the oscillations, which is similar to the Bernoulli map, power spectra, local Lyapunov exponents. It is possible to obtain two different scenarios of the SW-like attractor formation: via the quasiperiodicity destruction and via the transformation of the Feigenbaum chaotic attractor.

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Dynamics of ensemble of active Brownian particles interacting via Morse potential forces

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Dynamics of an one-dimensional ring of active Brownian particles interacting via morse potential forces is under study. Nonlinear friction defined in sense of Rayleigh. Density of ensemble can be varied.

In case of low density the statistics of clusters formation is studied. The evolution of border of bifurcational transition between ordered and disordered states is considered in a parameter space, particularly with different quantity of particles.

In high density ensemble the inception of solitons and their steady states are studied.

In both cases the transformation of particles velocity distribution is analyzed in terms of ensemble density and other parameters.

Evidence of a strange nonchaotic attractor in the El Nino dynamics

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Basing on a mathematical idea about the so-called strange nonchaotic attractor in the quasi-periodically forced dynamical systems, re-analyses data are considered. It is found that the El Nino - Southern Oscillation (ENSO) is driven not only by the Sun-induced periodic heating (seasonal and $\sim$ 11.2-year sunspot cycle), but also by two more external periodicities (incommensurate to the annual period) associated with the $\sim$ 18.6-year lunar-solar nutation of the Earth rotation axis, and the 14-month Chandler wobble in the Earth's pole motion. Because of the incommensurability of their periods all three forces affect the system in inappropriate time moments. As a result, the ENSO time series look to be very complex (strange in mathematical terms). The power spectra of these series reveal numerous peaks located at the periods that are multiples of the above periodicities as well as at their sub- and super-harmonic. In spite of this strangeness, a mutual order seems to be inherent to these time series and their spectra. This order reveals itself in the existence of a scaling of the power spectrum peaks and respective rhythms in the ENSO dynamics that look like the power spectrum and dynamics of the strange nonchaotic attractor. It means there are no limits to forecast ENSO, in principle. In practice, it opens a possibility to forecast ENSO for several years ahead.

Fermi acceleration in billards with holes

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Fermi acceleration is the study of the dynamics of a point particle moving within a closed region (billiard) with oscillating walls [1–3]. It is well known that the average energy growth of an ensemble of such particles in an ergodic chaotic billiard is linear with the number of collisions and quadratic as a function of time [2, 3]. In [4–7], it was discovered that if the ergodicity of the frozen billiard is violated (multi-component accelerators), then the average energy growth is much faster, typically exponential in time. This ergodicity breaking can be created by pseudo-integrability [4, 8] and also by division of the billiard configuration space into disjoint pieces [5], see Fig. 16 b. However, many such systems are not closed [9] and allow for entry and exit of particles (e.g. in plasma physics [1]). In this paper, we consider time-dependent leaky billiards and provide estimates for energy gain for the two above mentioned classes of the accelerators: ergodic (like in Figs. 16 a,c) and multi-component (Figs. 16 b,d).

For particles of fixed kinetic energy $E_{in} = \frac{v_{in}^2}{2}$ (assuming unit mass), the incoming energy flow is proportional to $hv_{in}E_{in}$. Neglecting self-interactions, it can be shown that the net energy production by the accelerator per unit of time is given by:

$$G(E_{in}) = hv_{in}[E_{out}?E_{in}] \tag{27}$$

where E_{out} is the averaged value of the kinetic energy at the moment of exit for a particle that enters the accelerator with the energy E_{in} (we average over all possible initial angles and positions in the hole, as well as over the phase of the billiard oscillations at the entry moment). It is natural to assume that $\bar{N}\ \frac{S}{h}\ \frac{V}{Lh}$, $\bar{N}^2\ \left(\frac{V}{Lh}\right)^2$ where h is the size of the hole in the billiard boundary, S is the size of the entire billiard boundary, V is the volume occupied by the billiard, and L is the characteristic diameter of the billiard. This has been numerically verified.

For the case of ergodic accelerator, it can be shown that in the small hole limit

$$E_{out}?E_{in} = k\frac{\bar{u}^2}{2}\bar{N} = k\frac{V}{Lh}\frac{\bar{u}^2}{2} \tag{28}$$

for some coefficient k that may depend on the billiard shape and on the details of the protocol of the billiard wall oscillations. Using Eq. 28 and Eq. 27, we obtain that the energy gain in the ergodic case is given by

$$G(E_{in}) = kv_{in}\frac{V}{L}\frac{\bar{u}^2}{2} \tag{29}$$

and is clearly independent of the hole size h. Similarly, for the multi-component accelerator, we get

$$E_{out} - E_{in} = \frac{\bar{u}^2}{2}\left[k_1\frac{v_{in}T}{L}\bar{N} + k_2\left(\frac{\bar{u}T}{L}\right)^2\bar{N}^2\right] \tag{30}$$

where the new coefficients $k_{1,2}$ depend on the shape of the billiard and the protocol of the wall oscillation. Thus, the energy gain for this case is:

$$G(E_{in}) = v_{in}\frac{V}{L}\frac{\bar{u}^2}{2}\left[k_1\frac{v_{in}T}{L}\bar{N} + k_2\mu T\frac{V}{Lh}\right] \tag{31}$$

On comparing Eq. 29 with Eq. 31, we clearly see that the gain rate G for the multi-component case can be made much larger than the gain in the ergodic case by diminishing the hole size or by increasing the incoming velocity. We have also verified our theoretical predictions by numerical simulations.

Dispersing accelerators (Figs. 16 a,b): at $t = 0$ a vertical bar is inserted at a position x_b, to the double Sinai billiard (a rectangle with two discs). Then the bar moves to the right with a constant velocity u till time T and then the bar is removed. The cycle restarts at time T. We consider two cases: 1a) ergodic case where the bar does not fully divide the billiard, covering 90% of the rectangle height (this case is called "Sinai"); 1b) multi-component case where the bar completely divides the billiard into two parts (hereafter "divided Sinai"). As mentioned earlier, these accelerators exhibit exponential-in-time energy growth in the multi-component case and quadratic-in-time energy growth in the ergodic case [5]. To examine the leaky behavior, two holes of length h are placed on the upper rectangle boundary.

Focusing accelerators (Figs. 16 c,d): The mushroom is a multi-component system having an integrable component and a chaotic component whereas the slanted stadium is fully ergodic and mixing. The oscillating mushroom accelerator exhibits exponential-in-time energy growth [7] whereas the oscillating stadium exhibits quadratic-in-time energy growth [2, 3].

In each numerical experiment 2000 particles are injected into the billiard through the holes, with random position in the hole, random entering angle and random initial phase (w.r.t. the oscillating walls). Each particle moves inside the billiard undergoing elastic collisions with the boundary till it exits by colliding with the hole. The quantity of primary importance is the energy of the particle at exit time.

Figure 17 shows the dependence of $G/v_{in} = (E_{out}?E_{in})h$ on $1/h$ for the four billiard types. Figure 17 a shows that the net energy flow increases linearly with $1/h$ for billiards with exponential acceleration (non-ergodic case) as predicted in Eq. 31. The inset shows that for sufficiently small holes the flow is essentially independent of the hole size for the ergodic billiards as predicted by Eq. 29.

It is important to note that for the leaky case, a faster accelerator does not automatically imply higher energy gain as faster particles escape earlier from the accelerator. The net energy gain reflects the balance between the averaged escape time and the energy gained by then. In this paper, we show that in a leaky billiard, the escape is mainly determined by the average number of collisions, and this essentially leads to the balance between these two factors for two different types of accelerators. Our results conclusively show that, for the multi-component case, the energy gain increases significantly when the hole size decreases whereas the energy gain is independent of the hole size for the ergodic case. For further details, please see [10].

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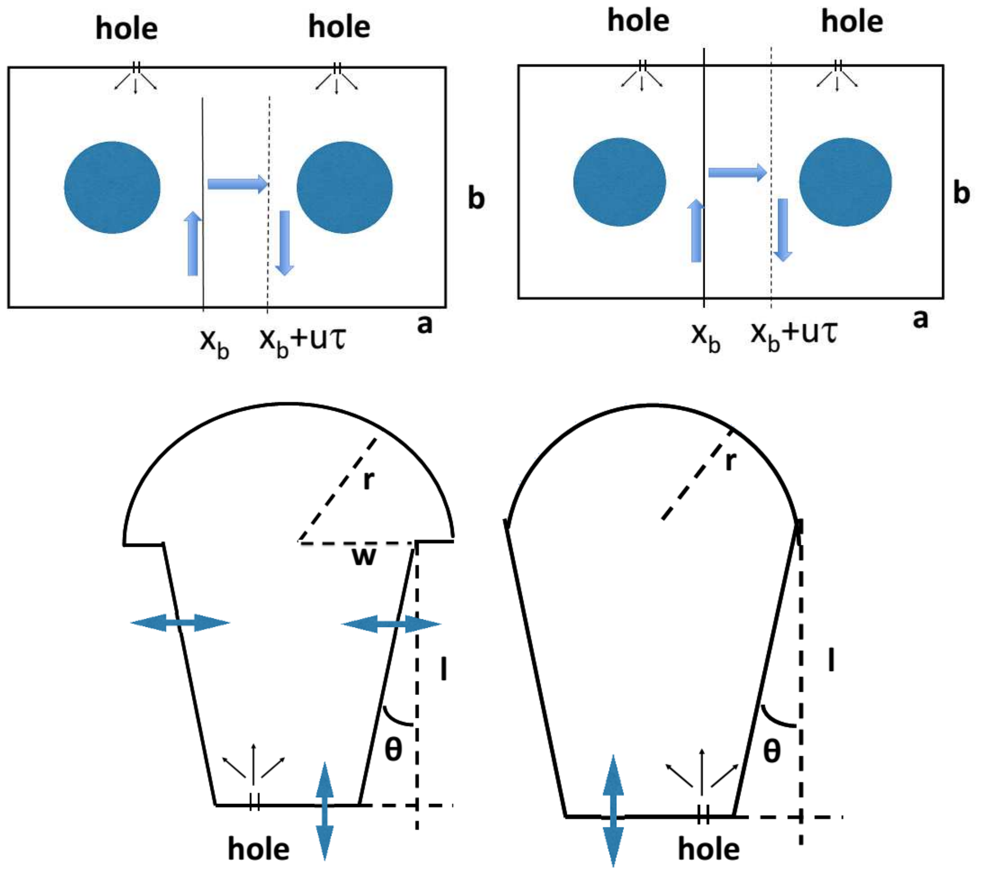

Figure 16: (a) Sinai accelerator; (b) divided Sinai accelerator; (c) mushroom accelerator; (d) stadium accelerator.

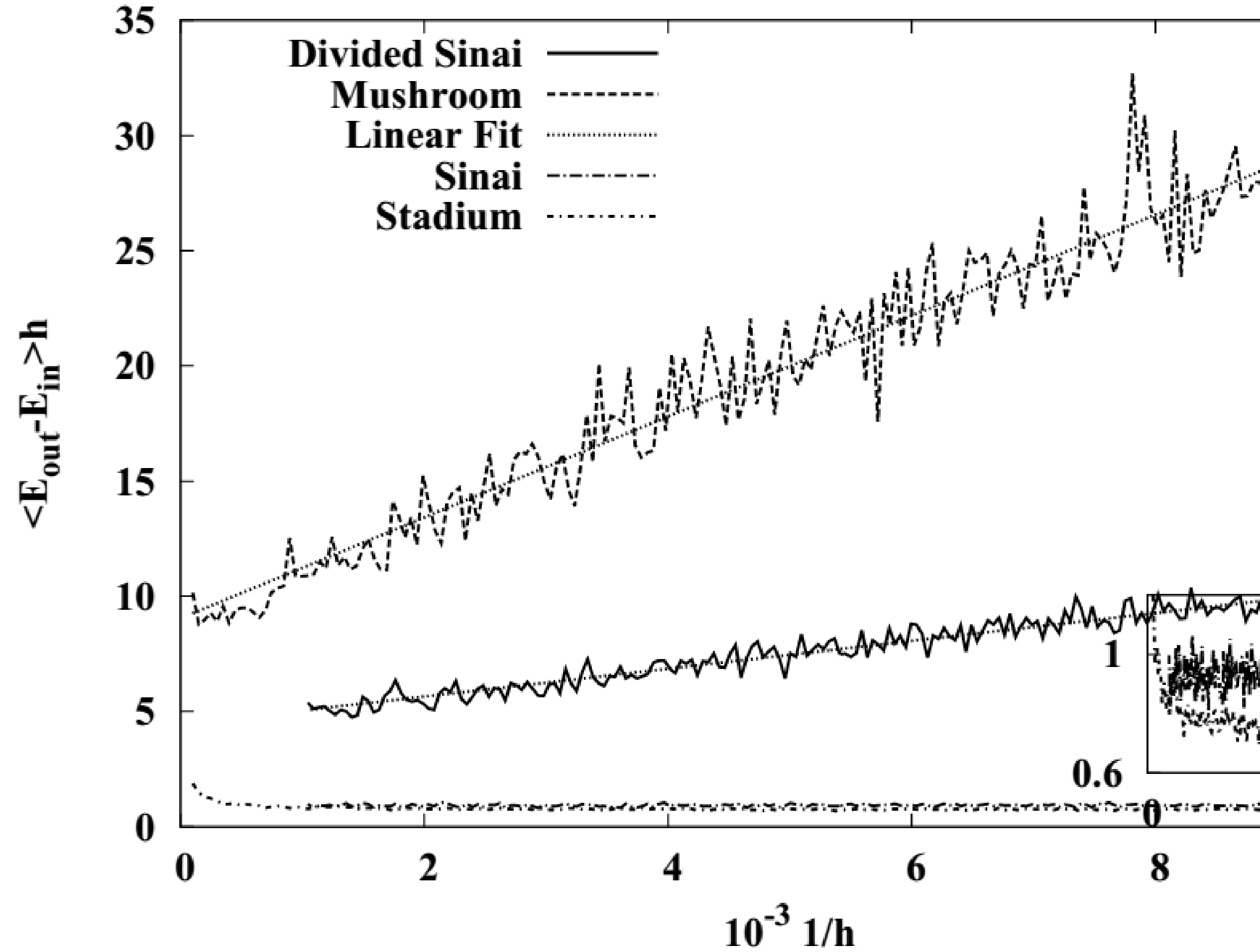

Figure 17: Averaged energy gain dependence on the hole size. The energy gain increases linearly with $1/h$ for multicomponent billiards, see Eq. 31, and is independent of the hole size for the ergodic billiards (for sufficiently small holes), see Eq. 29.

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Dynamical chimeras in a ring of oscillators with local coupling

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Chimera states in oscillatory ensembles are of great interest for the last time [1,2]. The typical feature of chimera regimes is nonlocal character of coupling of elements in ensemble. Beside of chimeras in ensembles the so named virtual chimeras can exist in the single systems with time delayed feedback [3]. It is known about the analogy between the system with delayed feedback and the spatially distributed system with periodic boundary conditions. Using this analogy one can obtain the dynamical chimera in the ring of oscillators with the local connections between the elements. Its properties should be the same as that of the chimera in the system with delayed feedback. In this work we construct the chimera similar to the described in [3] arising in the ring of oscillators with the local unidirectional nonlinear coupling. Beside of this we studied the chimera-like regimes in a ring of classical Du?ng?s oscillators with the simple local unidirectional coupling. The evolution of the traveling waves and the chimera-liked structures are investigated with the variation of the coupling strength.

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Lyapunov unstable Milnor attractors

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The *Milnor attractor* of a dynamical system is the smallest closed set that attracts almost every point. This notion of a global attractor is well defined for homeomorphisms of a compact metric measure space but the talk will be restricted to diffeomorphisms of smooth Riemannian manifolds.

Recall the classical notion of *Lyapunov stability*: a subset of the phase space is stable if for any its neighbourhood there is a smaller neighbourhood such that the orbits which start at the latter never quit the first one. It is easy to give an example of a diffeomorphism whose Milnor attractor is Lyapunov unstable and therefore it is only natural to ask how typical this instability of attractors can be.

It turns out that it is a locally topologically generic phenomenon, i.e., there are open domains in the space of diffeomorphisms such that residual (Baire 2^{nd} category) subsets of these domains consist of diffeomorphisms with Lyapunov unstable Milnor attractors. The crucial ingredient here is the existence of so-called *Newhouse domains*, open subsets of the space of diffeomorphisms where the maps exhibiting a homoclinic tangency associated with a continuation of a single hyperbolic saddle are dense and where coexistence of infinitely many sinks is generic. Whenever there is a Newhouse domain where the tangencies are associated with a *sectionally dissipative* saddle, a topologically generic diffeomorphism in this domain has an unstable Milnor atractor.

Another approach yields the following global statement. For a topologically generic C^1-diffeomorphism of a closed manifold, either any *homoclinic class* admits a *dominated splitting* (which is a very mild notion of hyperbolic-like behaviour) or the Milnor attractor is unstable for this diffeomorphism or for its inverse.

Global stabilisation for damped–driven conservation laws

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We consider a multidimensional conservation law with a damping term and a localised control. Our main result proves that any (non-stationary) solution $u(t,x)$ can be exponentially stabilised in the following sense: for any initial state one can find a control such that the difference between the corresponding solution and the function $u(t)$ goes to zero exponentially fast in an appropriate norm. As a consequence, we prove global exact controllability to solutions of the problem in question. We also establish global approximate controllability to solutions with the help of low-dimensional localised controls.

This is joint work with S. Rodrigues.

Chimera States in Ensemble of Non-Locally Coupled Anishchenko – Astakhov Self-Sustained Oscillators

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The ring of non-locally coupled Anishchenko – Astakhov self-sustained oscillators is under study. The equations which are described the ring are following:

$$
\begin{cases}
\dfrac{dx_i}{dt} &= mx_i + y_i - x_i z_i + \dfrac{\sigma}{2P}\sum\limits_{j=i-P}^{i+P}(x_j - x_i), \\
\dfrac{dy_i}{dt} &= -x_i + \dfrac{\sigma}{2P}\sum\limits_{j=i-P}^{i+P}(y_j - y_i), \\
\dfrac{dz_i}{dt} &= g(\Phi(x_i) - z_i),
\end{cases}
\\
\Phi(x) = \frac{x}{2}(x + |x|), \\
i = 1,\ 2,\ \ldots,\ N, \tag{32}
$$

where m and g — control parameters, σ — coupling strength, P — number of coupled neighbors, N — number of elements in a ring.

In a regime of weak chaos we demonstrate the existence of chimera states, traveling waves and various synchronization regimes in dependence on chosen initial conditions and parameters values. The reasons of occurrence of one or another regime are under discussion.

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Falling Motion of a Circular Cylinder Interacting Dynamically with Point Vortices

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The problem of falling motion of a body in fluid has a long history and was considered in a series of the classical and modern papers. Some of the effects described in the papers, such as periodic rotation (tumbling), can be encountered only in viscous fluids and thus demand for their proper treatment the use of the Navier - Stokes equations with boundary conditions specified on the body's surface. As a rule, such problems are hardly amenable to analytical analysis and can be addressed only numerically.

Another approach is to use (instead of the exact Navier - Stokes equations) some phenomenological ODE models which capture the viscous effects qualitatively.

In this paper we study the influence of the vorticity on the falling body in a trivial setting: a body (circular cylinder) subject to gravity is interacting dynamically with N point vortices. The circulation around the cylinder is not necessarily zero. So the model we consider here is exact and, at the same time, not so despairingly complex as most of the existing models are. The dynamical behavior of a heavy circular cylinder and N point vortices in an unbounded volume of ideal liquid is considered. The liquid is assumed to be irrotational and at rest at infinity. The circulation about the cylinder is different from zero. The governing equations are presented in Hamiltonian form. Integrals of motion are found. Allowable types of trajectories are discussed in the case of single vortex. The stability of finding equilibrium solutions is investigated and some remarkable types of partial solutions of the system are presented. Poincare sections of the system demonstrate chaotic behavior of dynamics, which indicates a non-integrability of the system.

In a case of zero circulation using autonomous integral we can also reduce the order of the system by one degree of freedom. Unlike nonzero circulation and the absence of vortices when the cylinder moves inside a certain horizontal stripe it is shown that in a presence of vortices and with circulation equal to zero vertical coordinate of the cylinder is unbounded decreasing. We then focus on the numerical study of dynamics of our system. In a case of zero circulation trajectories are noncompact. The different kinds of the scattering function of the vortex by cylinder were obtained. The form of these functions argues to chaotic behavior of the scattering which means that an additional analytical integral is absent.

An autonomous systems with quasiperiodic dynamics: examples and properties

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A traditional examples of a systems with quasiperiodic oscillations are non-autonomous systems. However, autonomous quasiperiodicity is possible in some generators of quasiperiodic oscillations. One of the example of this kind system is well-known Chua circuit [1]. Another example, is one of modification of the Lorenz system so called Lorenz-84 [2,3]. For this system invariant torus occurs on base of cycle of period 2. Prof. Anishchenko with coauthors suggest four-dimensional system in form of modification of generator of Anishenko-Astakhov [4]. This system also demonstrates bifurcation of torus doubling. In more detail is considered a new model of three-dimensional autonomous generator based on oscillator with hard excitation, which was suggested by prof. Kuznetsov, and its modification [5-7]. A chart of dynamical regimes and of Lyapunov exponents for autonomous system is presented, the possibility of realizing of hidden-attractors and main scenarios of destroying tori is discussed. A picture of synchronization for this system with external action is considered, in particular, Arnold resonance web was revealed in this system [8]. A structure of parameter plane for coupled generator of quasiperiodic oscillations is discussed. Results of experimental studying is presented.

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Invariant curves of quasi-periodically forced maps

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This talk deals with invariant curves of quasiperiodically forced smooth maps depending on a parameter,

$$\left.\begin{array}{rcl} \tilde{x} & = & f(x,\theta,\mu), \\ \tilde{\theta} & = & \theta+\omega \end{array}\right\}.$$

where $x \in R^n$, $\theta \in T$, ω is Diophantine and μ is a real parameter. The map f is assumed to be of class C^r, $r \geq 1$. An invariant curve is a C^1 map $\theta \mapsto x(\theta)$ such that $f(x(\theta),\theta,\mu) = x(\theta+\omega)$. Suppose that for one value of the parameter $\mu = \mu_0$, we have an attracting invariant curve of the system. We are interested in the behaviour of this curve when its Lyapunov exponent tends to zero when the parameter tends to some critical value μ_1. In particular, we want to study the fractalization phenomenon that might give rise to the appearence of Strange Non-Chaotic Attractors. In order to do this, we will focus on the most simple non-trivial situation, quasiperiodically forced affine systems in the plane. We will show that these systems have invariant curves that displays a fractalization process when the parameter tends to a critical value, if the corresponding linear system is non-reducible.

This is a joint work with A. Jorba, N. Fagella and M. Jorba-Cusco.

On Bonatti-Diaz cycles

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Piecewise contractions as models of regulatory networks

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Genetic regulatory networks are usually modeled by systems of coupled differential equations, and more particularly by systems of piecewise affine differential equations. Finite state models, better known as logical networks, are also used (see [1] and references therein). During the last years we have been studied a class of models which can be situated in the middle of the spectrum; they present both discrete and continuous aspects. They consist of a network of units, whose states are quantified by a continuous real variable. The state of each of these units evolves according to a contractive transformation chosen from a finite collection. The particular transformation chosen at each time step depends on the state of the neighboring units. In this way we obtain a network of coupled contractions. In this talk I will present some of the our theoretical results and biological applications which can be grouped in three cathegories as follows.

Dynamical Complexity Our first task has been the qualitatively description of the regulatory dynamics, with the aim of establishing relations between the dynamical complexity of the system and structural complexity of the underlying graph [2]. The dynamical complexity is a well-studied notion in the framework of the theory of dynamical systems, and it is related to the proliferation of distinguishable temporal behaviors. Following Kruglikov and Rypdal [3], it can be shown that the dynamical complexity of the regulatory dynamics grows subexponentially at the most, which implies that the behavior of the system is typically asymptotically periodic [2]. It also implies that the attractor of the system is a Cantor set [3]. In this respect, we have proven in [6] that for sufficiently strong contractions, the dynamical complexity of the system is polynomial, with a degree which can be associated to the structure of the underlying graph.

Dominant Vertices The dissipative and interdependent nature of the regulatory dynamics allows a size reduction of the system which we have studied in [7]. In that work we show that the knowledge of a trajectory on well chosen subcolections of vertices allows to determine the asymptotic dynamics of the whole network. We call the nodes in these distiguished subcollections, dominant vertices, and we completely characterize them from combinatorial grounds. We also propose an heuristic algorithm to compute those subcolletions of nodes, which we call dominant sets. Dominant sets have been used as a tool to clasify biological networks [8], and in principle could be used as strategic control sites.

Modularity In [2] we determine conditions under which the restriction of the dynamics on a subnetwork is equivalent to the dynamics one would observe in the subnetwork considered as an autonomous dynamical systems. We also have studied [5] the dynamical response of a small subnetwork subject to the action of the rest of the system, considering the former one as an open system under external inputs. Those two studies constitute a first rigorous approach to the notion of dynamical modularity.

Our work opens several interesting lines of theoretical and applied research that I will point out in this talk.

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One of the The main motivation of this work is to find explicit relations between the topological structure of the regulatory network and the growth rate of the dynamical complexity. In this article we derive general upper bounds for the dynamical complexity for networks of arbitrary size, and we exhibit specific instances of constraints imposed on the complexity growth by the structure of the underlying network.

Concerinig the dynamical complexity of regulatory dynamics several problems remain to be solved, amongst which the following:

- Relation to billiards.
- Structure of the atractors in the general case.
- Complexity in the case of “chaotic” and “critical” networks.

Concerinig the structure of dominant sets and its relation to the asymptotic dynamics, several tasks last to be done, amongst which the following:

- Design and characterization of algorithms to compute optimal minimal sets.
- Characterization of minimal dominant sets in the case of biological networks.
- Control design.

During the last years we have studied this class of piecewise contrating models of regulation, and have obtained a series of theoretical results and developed some very concrete biological applications.

Synchronization of traveling waves in the active medium with periodic boundary conditions

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The model of a one-dimensional active medium, which cells are the FitzHugh-Nagumo oscillators, is studied with periodical boundary conditions. Such medium has three different regimes in dependence on the parameters values. These regimes correspond to the self-sustained oscillations, excitable dynamics or bistability of the medium cells. Periodic boundary conditions provide the existence of traveling wave modes in all mention cases without any deterministic or stochastic excitation. The local and distributed priodic force influence on the medium are studied. The phenomena of the traveling wafrequency locking are found in all three regimes of active medium. The comparison synchronization effects in self-oscillatory, excitable and bistable regimes of the activmedium is fulflled.

The reported study was partially supported by RFBR, research project No. 14-52-12002, and by the Russian Ministry of Education and Science (project code 1008).

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Periodic forcing of a 2-dof Hamiltonian undergoing a Hamiltonian-Hopf bifurcation

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We consider a 2-dof Hamiltonian undergoing a Hamiltonian-Hopf bifurcation. We will review several aspects of this bifurcation. Our goal is to study the effect of a periodic perturbation acting on the previous Hamiltonian. Particular interest will be focused on the behaviour of the splitting of the invariant manifolds of the complex-saddle. Several aspects (normal forms, Poincaré maps, existence of invariant tori, spliting of the invariant manifolds, chaos, return maps,...) will be analysed as a combination of theoretical and numerical tools.

This is part of an ongoing work with E. Fontich and C. Simo.

New invariants of topological conjugacy of non-invertible inner mappings

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Let $f : X \to X$ be an inner surjective map of a locally compact locally connected metric space X. Recall that an inner map is an open and isolated map. A map is open if the image of an open set is open. A map is isolated if the pre-image of a point consists of isolated points.

The papers [2, 3]introduced a set of new invariants of topological conjugacy of non-invertible inner mappings that are modeled from the invariant sets of dynamical systems generated by homeomorphisms. Those new invariants are based on the analogy between the trajectories of a homeomorphism and the directions in the set of points having common image which is viewed as having 2 dimensions.

In particular, this papers introduced the sets of neutrally recurrent and the neutrally non-wandering points related to the dynamics of points and neighborhoods in that "extra" dimension. Those invariants provide a natural language for the topological classification of many classes of polynomial maps and also allow to define analogs of many well known classes of invertible maps such as Smale diffeomorphisms for the non-invertible inner maps.

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